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ABSTRACT

Ebbeler, Jr., Donald Herman. Ph.D., Purdue University, August, 1970. An Investigation of the Properties of the Exact Finite Sample Distributions of GCL Statistics Associated with the Structural Representation of an Econometric Model. Major Professor: Robert L. Basmann.

This dissertation is a study of the properties of the distribution functions of the GCL identifiability test statistic, endogenous variable structural coefficient estimator, and structural variance estimators for an econometric model with two endogenous variables included in the equation of interest.

Chapter 1 contains a short presentation of the simultaneous equations representation of the econometric model and some standardizing transformations which simplify the expressions for the distribution functions of the GCL statistics investigated.

Chapter 2 contains a derivation of an approximation to the distribution function for the GCL endogenous variable structural coefficient estimator and some tabulations of the approximating distribucion function.

Chapter 3 contains a demonstration that the distribution function of the GCL identifiability test statistic has a corresponding (central) F asymptotic distribution function. The method of moments is used to determine parameters of the (central) F distribution used to approximate the exact distribution function and some tabulations of the approximating distribution function are presented.

Chapter 4 contains a demonstration that the distribution functions of the three GCL structural variance estimators have corresponding (central) χ^2 asymptotic distribution functions.

Chapter 5 contains a derivation of approximations of the distribution functions of the three GCL structural variance estimators. The use of the method of moments to determine a gamma distribution which can be used to approximate an exact distribution function is also discussed.

Chapter 6 contains some tabulations of the distribution functions of the three GCL structural variance estimators.

The appendix contains definitions and theorems from special function theory which are of value in the demonstrations presented in this dissertation.

INTRODUCTION

One class of explanatory econometric models is the simultaneous equations model. To test such a model it is necessary to compute confidence intervals for the various estimators and test statistics associated with the model. For the Two Stage Least Squares variant of the Generalized Classical Linear (GCL) Least Variance Difference principle of estimation Basmann and Richardson have derived the exact finite sample distribution functions for several of the estimators and test statistics associated with a simultaneous equations econometric model for which the equation of interest contains two endogenous variables.

In each exact finite sample distribution function there appears the parameter $\bar{\mu}^2$, called the concentration parameter, which is a quadratic form containing the exogenous variable sample observations and the structural parameters. The name 'concentration parameter' derives from the fact that the GCL estimator of each structural parameter of the equation of interest converges in probability to the corresponding population structural parameter as $\bar{\mu}^2 \rightarrow \infty$, sample size being fixed (Basmann and Richardson, 1969a, p. 2).

In the dissertation we investigate properties of the exact finite sample distribution functions of the GCL identifiability test statistic, GCL endogenous variable structural coefficient estimator, and the three GCL structural variance estimators for purpose of determining expressions or approximations for the exact distribution functions when $\bar{\mu}^2 \to \infty$. We also tabulate some of the exact distribution functions, investigate methods of their approximation, and for the three GCL structural variance estimators we develop algorithms under which we may specify regions of the parameter space associated with a particular exact distribution function in which the exact distribution function is acceptably approximated by its corresponding asymptotic distribution function.

In Chapter 1 we present the model and some simplifying standardizing transformations introduced by Basmann. In Chapter 2 we derive an approximation to the distribution function for the GCL endogenous variable structural coefficient estimator by a method investigated by Basmann (Basmann, 1963b, pp. 2-3) and we present some tabulations of the approximating distribution function. In Chapter 3 we demonstrate that the distribution function of the GCL identifiability test statistic has a corresponding (central) F asymptotic distribution function and we tabulate an approximation to the exact distribution function by the method of moments. In Chapters 4, 5, and 6 we investigate the exact finite sample distribution functions of the three GCL structural variance estimators; in Chapter 4 we demonstrate that the corresponding asymptotic distribution functions are (central) χ^2; in Chapter 5 we derive expressions for approximations to the exact distribution functions and discuss the use of the method of moments to approximate the exact distribution functions; in Chapter 6 we present tabulations of the exact distribution functions and the various approximations. In the appendix we present definitions and theorems from special function theory which proved useful in the dissertation. Some original theorems involving higher order confluent hypergeometric series which were used in Chapters 3 and 4 are included in the appendix.

CHAPTER 1

AN ECONOMETRIC MODEL

1.1 The Simultaneous Equations Model

Let

$$(1.1.1) \quad - y_{t1} + \beta_1 y_{t2} + \sum_{j=1}^{K_1} \gamma_j z_{tj} + e_{t1} = 0$$

be the first structural equation from the set of nondynamic stochastic simultaneous equations

$$(1.1.2) \quad B'y'_{t.} + \Gamma'z'_{t.} + e'_{t.} = 0 \qquad t = 1, 2, \ldots, N$$

where B is a real nonsingular matrix with dimensions G x G, $G \geqq 2$ and Γ is a real matrix with dimensions K x G, $N > K \geqq K_1 + 1$. $y'_{t.}$ and $z'_{t.}$ are conformable vectors of G endogenous and K exogenous variables, respectively. The G-dimensional vectors $e_{t.}$ are assumed to be independently distributed as the multivariate normal with a positive definite symmetric G x G covariance matrix Ω and a zero mean vector.

$$(1.1.3) \quad f(e_{t.}) = (2\pi)^{-\frac{G}{2}} |\Omega|^{-\frac{1}{2}} e^{-\frac{1}{2} e_{t.} \Omega^{-1} e'_{t.}}$$

$$t = 1, 2, \ldots, N$$

Since B is nonsingular, for each ordered set (B, Γ, Ω) there exists a unique corresponding ordered set (I_G, Π, Σ) which is said to be <u>observationally equivalent</u> to (B, Γ, Ω) (Basmann, 1965, pp. 1083, 1087-1089) where Π and Σ are specified by

$$\text{(1.1.4 a-b)} \quad \Pi = - \Gamma B^{-1}$$

$$\Sigma = (B')^{-1} \, \Omega \, B^{-1}$$

and the <u>reduced-form model</u> observationally equivalent to the <u>structural model</u> specified by (1.1.2) and (1.1.3) is given by

$$\text{(1.1.5)} \quad y'_{t.} = \Pi' z'_{t.} + \eta'_{t.}$$

$$t = 1, 2, \ldots, N$$

$$\text{(1.1.6)} \quad g(\eta_{t.}) = (2\pi)^{-\frac{G}{2}} \, |\Sigma|^{-\frac{1}{2}} \, e^{-\frac{1}{2} \eta_{t.} \Sigma^{-1} \eta'_{t.}}$$

$$t = 1, 2, \ldots, N$$

where $\eta_{t.}$ is specified by

$$\text{(1.1.7)} \quad \eta'_{t.} = - (B')^{-1} \, e'_{t.}$$

Let

$$\text{(1.1.8 a-c)} \quad Y = [y_{ti}], \; t = 1, \ldots, N; \; i = 1, \ldots, G$$

$$Z = [z_{tk}],\ t = 1, \ldots, N;\ k = 1, \ldots, K$$

$$Z_1 = [z_{tk}],\ t = 1, \ldots, N;\ k = 1, \ldots, K_1$$

We will require that exactly one column of Z have all N entries be 1. $\beta_{.1}$ is a G-dimensional vector defined by

(1.1.9) $\beta'_{.1} = (-1, \beta_1, 0, \ldots, 0)$

and ν is defined by

(1.1.10) $\nu = K - K_1 - 1 \geqq 0$

Let Z_2 be defined by

(1.1.11) $Z = [Z_1 \vdots Z_2]$

and let Π be partitioned conformably.

(1.1.12) $\Pi' = [\Pi_1' \vdots \Pi_2']$

Designate the second row of Π_2' by Π_{22}' and let the elements of Ω and Σ be defined by

(1.1.13 a-b) $\Omega = [\omega_{ij}],\ i, j, = 1, \ldots, G$

$$\Sigma = [\sigma_{ij}],\ i,\ j = 1,\ \ldots,\ G$$

The <u>concentration parameter</u> associated with (1.1.1) is defined by

$$(1.1.14)\quad \bar{\mu}^2 = \sigma_{22}^{-1}\ \Pi_{22}'\ Z_2'\ [I_N - Z_1(Z_1'Z_1)^{-1}\ Z_1']Z_2\ \Pi_{22}$$

(Basmann, 1963a, pp. 966-967; Richardson, 1968b, p. 1219).

1.2 Standardizing Transformations for Associated GCL Statistics

In the process of deriving the exact finite sample distribution functions of GCL statistics associated with (1.1.2) it is convenient to introduce <u>standardizing transformations</u> in order to decrease the the notational burden (Basmann, 1963a, p. 966; Richardson, 1968b, p. 1216; Schoepfle, 1969, Chapter 3; McDonald, 1970, Chapter 1).

$\hat{\beta}_1$ is defined by the G-dimensional GCL estimator of $\beta_{.1}$, $\hat{\beta}_{.1}$.

$$(1.2.1)\quad \hat{\beta}_{.1} = (-1,\ \hat{\beta}_1,\ 0,\ \ldots,\ 0)$$

For $\nu > 0$, GCL estimation of structural parameters admits three alternative classes of estimators of ω_{11}. Each class of estimators corresponds to exactly one of the quadratic forms

$$(1.2.2\ \text{a-c})\quad G_1(\beta_{.1}) = \beta_{.1}'\ Y'[I_N - Z_1(Z_1'Z_1)^{-1}\ Z_1']Y\ \beta_{.1}$$

$$G_2(\beta_{.1}) = \beta'_{.1}Y'[I_N - Z(Z'Z)^{-1} Z']Y \beta_{.1}$$

$$Q(\beta_{.1}) = G_1(\beta_{.1}) - G_2(\beta_{.1})$$

(Basmann and Richardson, 1969b, p. 3). GCL estimators of ω_{11} are specified by

$$\hat{\omega}_{11} = \frac{1}{N} G_1(\hat{\beta}_{.1})$$

$$(1.2.3\ \text{a-c}) \quad \tilde{\omega}_{11} = \frac{1}{N} G_2(\hat{\beta}_{.1})$$

$$\bar{\omega}_{11} = \frac{1}{\nu} Q(\hat{\beta}_{.1})$$

(Basmann and Richardson, 1969b, p. 3). The variable whose distribution function is computed in each of (1.2.3 a-c) is designated by U and is respectively defined by

$$U = \frac{N\hat{\omega}_{11}}{\omega_{11}}$$

$$(1.2.4\ \text{a-c}) \quad U = \frac{N\tilde{\omega}_{11}}{\omega_{11}}$$

$$U = \frac{\nu\bar{\omega}_{11}}{\omega_{11}}$$

Standardizing transformations provide the following definitions of $\bar{\beta}_1$, V_1, V_2, V_3, and V_4:

$$\beta_1 = \sigma_{22}^{-1}\left[\sigma_{12} + \bar{\beta}_1(\sigma_{11}\sigma_{22} - \sigma_{12}^2)^{\frac{1}{2}}\right]$$

$$\hat{\beta}_1 = \sigma_{22}^{-1}\left[\sigma_{12} + V_1(\sigma_{11}\sigma_{22} - \sigma_{12}^2)^{\frac{1}{2}}\right]$$

(1.2.5 a-e)

$$Q(\hat{\beta}_{.1}) = \sigma_{22}^{-1} V_2(\sigma_{11}\sigma_{22} - \sigma_{12}^2)$$

$$G_2(\hat{\beta}_{.1}) = \sigma_{22}^{-1} V_3(\sigma_{11}\sigma_{22} - \sigma_{12}^2)$$

$$G_1(\hat{\beta}_{.1}) = \sigma_{22}^{-1} V_4(\sigma_{11}\sigma_{22} - \sigma_{12}^2)$$

From (1.1.4 a-b), (1.2.3 a-c), (1.2.4 a-c), and (1.2.5 a-e) we deduce the following relationships corresponding to the three definitions of (1.2.4 a-c):

$$U = V_4/(1 + \bar{\beta}_1^{\,2})$$

(1.2.6 a-c) $U = V_3/(1 + \bar{\beta}_1^{\,2})$

$$U = V_2/(1 + \bar{\beta}_1^{\,2})$$

CHAPTER 2

THE DISTRIBUTION FUNCTION FOR THE GCL ENDOGENOUS VARIABLE STRUCTURAL COEFFICIENT ESTIMATOR

2.1 An Approximation of the Distribution Function

Richardson (1968b) has derived the exact finite sample density function corresponding, in the way specified by (1.2.5b), to the GCL endogenous variable structural coefficient estimator $\hat{\beta}_1$.

$$(2.1.1) \quad h(V_1) = \frac{\Gamma\left(\frac{\nu+2}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{\nu+1}{2}\right)} \cdot \frac{e^{-\frac{\bar{\mu}^2}{2}}}{\left(1 + V_1^2\right)^{\frac{\nu+2}{2}}}$$

$$\times \sum_{j=0}^{\infty} \frac{\left(\frac{\nu+2}{2}\right)_j \left(\frac{\bar{\mu}^2}{2}\right)^j}{\left(\frac{\nu+1}{2}\right)_j \quad j!} \left[\frac{(1+\bar{\beta}_1 V_1)^2}{1 + V_1^2}\right]^j$$

$$\times \; {}_1F_1\left(\frac{1}{2} + j;\; \frac{\nu+1}{2} + j;\; -\frac{\bar{\beta}_1^2 \bar{\mu}^2}{2}\right)$$

$$-\infty < V_1 < \infty$$

(Richardson, 1968b, pp. 1218-1219).

We define λ^2, z, x_1, x, and $A_m(z)$ by

$$\lambda^2 = \bar{\mu}^2(1 + \bar{\beta}_1^{\,2})$$

$$z = \frac{\bar{\mu}^2\bar{\beta}_1^{\,2}}{4}$$

(2.1.2 a-e) $$x_1 = \frac{\bar{\mu}(\bar{\beta}_1 V_1 + 1)}{(1 + V_1^{\,2})^{1/2}}$$

$$x = \frac{\bar{\mu}(V_1 - \bar{\beta}_1)}{(1 + V_1^{\,2})^{1/2}}$$

$$A_m(z) = \frac{e^z\,\Gamma\left(\frac{\nu+2}{2} + m\right)}{m!\,\Gamma\left(\frac{\nu+1}{2} + m\right)}$$

$$\times\ {}_1F_1\left(\frac{1}{2} + m;\ \frac{\nu+1}{2} + m;\ - 2z\right)$$

(Basmann, 1963b, p. 2).

Then from (2.1.2) we obtain

(2.1.3 a-b) $$dx = \frac{\bar{\mu}(\bar{\beta}_1 V_1 + 1)}{(1 + V_1^{\,2})^{3/2}}\ dV_1$$

$$x_1^{\,2} = \lambda^2 - x^2$$

From (2.1.1), (2.1.2), and (2.1.3) we obtain

$$(2.1.4) \quad h(V_1) = \frac{e^{-\frac{\lambda^2}{2} + z}}{\Gamma\left(\frac{1}{2}\right)\left(1 + V_1^{\,2}\right)^{\frac{\nu+2}{2}}}$$

$$\times \sum_{m=0}^{\infty} A_m(z) \left(\frac{\lambda^2 - x^2}{2}\right)^m$$

$$- \infty < V_1 < \infty$$

Making use of the <u>binomial</u> <u>formula</u> (Hall and Knight, 1950, p. 138) we write (2.1.4) as

$$(2.1.5) \quad h(V_1) = \frac{e^{-\frac{\lambda^2}{2} + z}}{\Gamma\left(\frac{1}{2}\right)\left(1 + V_1^{\,2}\right)^{\frac{\nu+2}{2}}}$$

$$\times \sum_{m=0}^{\infty} A_m(z) \sum_{n=0}^{m} \frac{m!}{n!\,(m-n)!} \left(\frac{\lambda^2}{2}\right)^{m-n} \left(\frac{-x^2}{2}\right)^n$$

$$- \infty < V_1 < \infty$$

By interchanging the order of summation (2.1.5) can be written as

$$(2.1.6) \quad h(V_1) = \frac{1}{\Gamma\left(\frac{1}{2}\right)\left(1 + V_1^{\,2}\right)^{\frac{\nu+2}{2}}}$$

$$\times \sum_{n=0}^{\infty} \left[e^{-\frac{\lambda^2}{2} + z} \sum_{m=n}^{\infty} \frac{m!\, A_m(z)}{(m-n)!} \left(\frac{\lambda^2}{2}\right)^{m-n} \right]$$

$$\times \frac{1}{n!} \left(-\frac{x^2}{2}\right)^n$$

$$-\infty < V_1 < \infty$$

We define C_n as

$$(2.1.7) \quad C_n = e^{-\frac{\lambda^2}{2} + z} \sum_{m=n}^{\infty} \frac{m!\, A_m(z)}{(m-n)!} \left(\frac{\lambda^2}{2}\right)^{m-n}$$

From (2.1.2), (2.1.7), and (A.13) we obtain, by replacing $\underline{m}$ by $\underline{m+n}$

$$(2.1.8) \quad C_n = e^{-\frac{\lambda^2}{2}} \frac{\Gamma\left(\frac{\nu+2}{2} + n\right)}{\Gamma\left(\frac{\nu+1}{2} + n\right)}$$

$$\times \sum_{m=0}^{\infty} \frac{\left(\frac{\nu+2}{2} + n\right)_m}{\left(\frac{\nu+1}{2} + n\right)_m} \cdot \frac{\left(\frac{\lambda^2}{2}\right)^m}{m!}$$

$$\times {}_1F_1\left(\frac{\nu}{2};\ \frac{\nu+1}{2} + n + m;\ 2z\right)$$

Replacing the <u>confluent</u> <u>hypergeometric</u> <u>function</u> in (2.1.8) by its series expansion using (A.4) and making use of (A.2) we obtain

$$(2.1.9)\quad C_n = e^{-\frac{\lambda^2}{2}} \frac{\Gamma\left(\frac{\nu+2}{2}+n\right)}{\Gamma\left(\frac{\nu+1}{2}+n\right)}$$

$$\times \sum_{m=0}^{\infty} \sum_{t=0}^{\infty} \frac{\left(\frac{\nu+2}{2}+n\right)_m \left(\frac{\nu}{2}\right)_t}{\left(\frac{\nu+1}{2}+n\right)_{m+t}} \cdot \frac{\left(\frac{\lambda^2}{2}\right)^m (2z)^t}{m!\ t!}$$

Replacing <u>m</u> by <u>m-t</u> in (2.1.9) and making use of (2.1.2) and (A.7) we obtain

$$(2.1.10)\quad C_n = e^{-\frac{\lambda^2}{2}} \frac{\Gamma\left(\frac{\nu+2}{2}+n\right)}{\Gamma\left(\frac{\nu+1}{2}+n\right)}$$

$$\times \sum_{m=0}^{\infty} \frac{\left(\frac{\nu+2}{2}+n\right)_m}{\left(\frac{\nu+1}{2}+n\right)_m} \cdot \frac{\left(\frac{\lambda^2}{2}\right)^m}{m!}$$

$$\times \sum_{t=0}^{\infty} \frac{(-m)_t \left(\frac{\nu}{2}\right)_t}{\left(-\frac{\nu}{2}-n-m\right)_t} \cdot \frac{\left(\frac{\bar{\beta}_1^{\,2}}{1+\bar{\beta}_1^{\,2}}\right)^t}{t!}$$

From (2.1.10), (A.4), and (A.12) we obtain

$$(2.1.11)\quad C_n = e^{-\frac{\lambda^2}{2}} (1+\bar{\beta}_1^{\,2})^{n+\nu} \frac{\Gamma\left(\frac{\nu+2}{2}+n\right)}{\Gamma\left(\frac{\nu+1}{2}+n\right)}$$

$$\times \sum_{m=0}^{\infty} \frac{\left(\frac{\nu+2}{2} + n\right)_m}{\left(\frac{\nu+1}{2} + n\right)_m} \cdot \frac{\left(\frac{\lambda^2}{2}\right)^m}{m!}$$

$$\times \sum_{t=0}^{\infty} \frac{(-\frac{\nu}{2} - n)_t\ (-\nu - n - m)_t}{(-\frac{\nu}{2} - n - m)_t} \cdot \frac{\left(\frac{\bar{\beta}_1^{\,2}}{1 + \bar{\beta}_1^{\,2}}\right)^t}{t!}$$

Using (A.2) and (A.7), (2.1.11) can be written

$$\text{(2.1.12)} \quad C_n = e^{-\frac{\lambda^2}{2}} (1 + \beta_1^{\,2})^{n+\nu} \quad \frac{\Gamma(\nu + 1 + n)}{\Gamma\left(\frac{\nu+1}{2} + n\right)}$$

$$\times \sum_{t=0}^{\infty} \frac{\left(-\frac{\nu}{2} - n\right)_t \Gamma\left(\frac{\nu+2}{2} + n - t\right)}{\Gamma(\nu + 1 + n - t)} \cdot \frac{\left(\frac{\bar{\beta}_1^{\,2}}{1 + \bar{\beta}_1^{\,2}}\right)^t}{t!}$$

$$\times \sum_{m=0}^{\infty} \frac{(\nu+1+n)_m \left(\frac{\nu+2}{2} + n - t\right)_m}{\left(\frac{\nu+1}{2} + n\right)_m (\nu + 1 + n - t)_m} \cdot \frac{\left(\frac{\lambda^2}{2}\right)^m}{m!}$$

From (A.4) and (A.18) we have

$$\text{(2.1.13)} \quad \sum_{m=0}^{\infty} \frac{(\nu+1+n)_m \left(\frac{\nu+2}{2} + n - t\right)_m}{\left(\frac{\nu+1}{2} + n\right)_m (\nu+1+n-t)_m} \cdot \frac{\left(\frac{\lambda^2}{2}\right)^m}{m!}$$

$$\rightarrow \frac{\Gamma\left(\frac{\nu+1}{2} + n\right)\Gamma(\nu+1+n-t)}{\Gamma(\nu+1+n)\Gamma\left(\frac{\nu+2}{2} + n - t\right)} \; e^{\frac{\lambda^2}{2}} \left(\frac{\lambda^2}{2}\right)^{\frac{1}{2}}$$

as $\lambda^2 \rightarrow \infty$.

From (2.1.2), (2.1.12), and (2.1.13) we have

$$\text{(2.1.14)} \quad c_n \rightarrow \left(\frac{\bar{\mu}^2}{2}\right)^{\frac{1}{2}} \; (1 + \bar{\beta}_1^{\,2})^{\frac{\nu+1}{2}}$$

as $\bar{\mu}^2 \rightarrow \infty$.

Then, from (2.1.6), (2.1.7), and (2.1.14) we have

$$\text{(2.1.15)} \quad h(V_1) \simeq \frac{\bar{\mu}\,(1 + \bar{\beta}_1^{\,2})^{\frac{\nu+1}{2}}}{\sqrt{2\pi}\,(1+V_1^{\,2})^{\frac{\nu+2}{2}}} \; e^{-\frac{x^2}{2}}$$

$$-\infty < V_1 < \infty$$

for large $\bar{\mu}^2$.

From (2.1.3) and (2.1.15) we have

$$\text{(2.1.16)} \quad h(V_1)dV_1 \simeq$$

$$\frac{(1 + \bar{\beta}_1^{\,2})^{\frac{\nu+1}{2}}}{(1+V_1^{\,2})^{\frac{\nu-1}{2}}\,(\bar{\beta}_1 V_1 + 1)} \cdot \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\,dx$$

$$- \infty < V_1 < \infty$$

for large $\bar{\mu}^2$.

Let the distribution function of V_1 be defined by

$$(2.1.17) \quad H(V_1) = \int_{-\infty}^{V_1} h(v_1)\, dv_1$$

$$- \infty < V_1 < \infty$$

Richardson (1968b, p. 1225) has demonstrated that

$$(2.1.18) \quad H(V_1) \rightarrow \begin{cases} 0 \text{ if } V_1 < \bar{\beta}_1 \\ 1 \text{ if } V_1 \geqq \bar{\beta}_1 \end{cases}$$

$$- \infty < V_1 < \infty$$

as $\bar{\mu}^2 \rightarrow \infty$.

Basmann (1963b) has obtained an approximation for $H(V_1)$ when $\nu=1$. Setting $\nu=1$ in (2.1.16) gives Basmann's expression for $h(V_1)dV_1$ for large $\bar{\mu}^2$. We will approximate $H(V_1)$ from (2.1.16) by means of <u>incomplete beta functions</u>, $I_B[a, b; x]$ (A.34).

Since the transformations (2.1.2 c,d) are not monotonic, care must be exercised in performing the integrations and making the changes of variables required to obtain the final form of our approximation of $H(V_1)$.

From (2.1.2), (2.1.16), and (2.1.17) we obtain

$$(2.1.19)\quad H(V_1) \simeq \int_{-\bar{\mu}}^{\frac{\bar{\mu}(V_1-\bar{\beta}_1)}{(1+V_1^2)^{1/2}}} \left[1-\left(\frac{x}{\lambda}\right)^2\right]^{\frac{\nu-1}{2}}$$

$$\times\ (1-\bar{\beta}_1\cdot\frac{x}{x_1})\ \left|1-\bar{\beta}_1\cdot\frac{x}{x_1}\right|^{\nu-1}\cdot\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\ dx$$

$$-\infty < V_1 < \infty$$

Making the substitution $(\frac{x}{\lambda})^2 = t$ we obtain the following expressions for the approximation of $H(V_1)$:

For $\bar{\beta}_1 = 0$

$$(2.1.20\ \text{a-b})\quad H(V_1) \simeq \frac{\bar{\mu}}{2\sqrt{2\pi}}$$

$$\times\left\{\int_0^1 t^{-\frac{1}{2}}(1-t)^{\frac{\nu-1}{2}} e^{-\frac{\bar{\mu}^2}{2}t}\,dt - \int_0^{\frac{V_1^2}{1+V_1^2}} t^{-\frac{1}{2}}(1-t)^{\frac{\nu-1}{2}} e^{-\frac{\bar{\mu}^2}{2}t}\,dt\right\}$$

$$-\infty < V_1 \leqq 0$$

$$H(V_1) \simeq \frac{\bar{\mu}}{2\sqrt{2\pi}}$$

$$\times \left\{ \int_0^1 t^{-\frac{1}{2}} (1 - t)^{\frac{\nu-1}{2}} e^{-\frac{\bar{\mu}^2}{2} t} dt \right.$$

$$\left. + \int_0^{\frac{V_1^2}{1+V_1^2}} t^{-\frac{1}{2}} (1 - t)^{\frac{\nu-1}{2}} e^{-\frac{\bar{\mu}^2}{2} t} dt \right\}$$

$$0 \leqq V_1 < \infty$$

For $\bar{\beta}_1 > 0$

(2.1.21 a-c) $$H(V_1) \simeq \frac{\lambda}{2\sqrt{2\pi}}$$

$$\times \left\{ \int_0^{\frac{1}{(1+\bar{\beta}_1^2)} \cdot \frac{(V_1-\bar{\beta}_1)^2}{(1+V_1^2)}} t^{-\frac{1}{2}} (1 - t)^{\frac{\nu-1}{2}} (-)^{\nu} \right.$$

$$\times \left[1 - |\bar{\beta}_1| \left(\frac{t}{1-t}\right)^{\frac{1}{2}}\right]^{\nu} e^{-\frac{\lambda^2}{2} t} dt$$

$$- \int_0^{\frac{1}{(1+\bar{\beta}_1^2)}} t^{-\frac{1}{2}} (1 - t)^{\frac{\nu-1}{2}} (-)^{\nu}$$

$$\left. \times \left[1 - |\bar{\beta}_1| \left(\frac{t}{1-t}\right)^{\frac{1}{2}}\right]^{\nu} e^{-\frac{\lambda^2}{2} t} dt \right\}$$

$$- \infty < V_1 \leq \frac{-1}{\bar{\beta}_1}$$

$$H(V_1) \simeq \frac{\lambda}{2\sqrt{2\pi}}$$

$$\times \left\{ \int_0^1 t^{-\frac{1}{2}} (1 - t)^{\frac{\nu-1}{2}} (-)^{\nu} \right.$$

$$\times \left[1 - |\bar{\beta}_1| \left(\frac{t}{1-t}\right)^{\frac{1}{2}}\right]^{\nu} e^{-\frac{\lambda^2}{2} t} dt$$

$$- \int_0^{\frac{1}{(1+\bar{\beta}_1^2)}} t^{-\frac{1}{2}} (1 - t)^{\frac{\nu-1}{2}} (-)^{\nu}$$

$$\times \left[1 - |\bar{\beta}_1| \left(\frac{t}{1-t}\right)^{\frac{1}{2}}\right]^{\nu} e^{-\frac{\lambda^2}{2} t} dt$$

$$+ \int_0^1 t^{-\frac{1}{2}} (1 - t)^{\frac{\nu-1}{2}}$$

$$\times \left[1 + |\bar{\beta}_1| \left(\frac{t}{1-t}\right)^{\frac{1}{2}}\right]^{\nu} e^{-\frac{\lambda^2}{2} t} dt$$

$$- \int_0^{\frac{1}{(1+\bar{\beta}_1^2)} \cdot \frac{(V_1-\bar{\beta}_1)^2}{(1 + V_1^2)}} t^{-\frac{1}{2}} (1 - t)^{\frac{\nu-1}{2}}$$

$$\times \left[1 + |\bar{\beta}_1| \left(\frac{t}{1-t}\right)^{\frac{1}{2}}\right]^{\nu} e^{-\frac{\lambda^2}{2} t} dt\Big\}$$

$$-\frac{1}{\bar{\beta}_1} \leqq V_1 \leqq \bar{\beta}_1$$

$$H(V_1) \simeq \frac{\lambda}{2\sqrt{2\pi}}$$

$$\times \Big\{ \int_0^1 t^{-\frac{1}{2}} (1 - t)^{\frac{\nu-1}{2}} (-)^{\nu}$$

$$\times \left[1 - |\bar{\beta}_1| \left(\frac{t}{1-t}\right)^{\frac{1}{2}}\right]^{\nu} e^{-\frac{\lambda^2}{2} t} dt$$

$$- \int_0^{\frac{1}{(1+\bar{\beta}_1^2)}} t^{-\frac{1}{2}} (1 - t)^{\frac{\nu-1}{2}} (-)^{\nu}$$

$$\times \left[1 - |\bar{\beta}_1| \left(\frac{t}{1-t}\right)^{\frac{1}{2}}\right]^{\nu} e^{-\frac{\lambda^2}{2} t} dt$$

$$+ \int_0^1 t^{-\frac{1}{2}} (1 - t)^{\frac{\nu-1}{2}}$$

$$\times \left[1 + |\bar{\beta}_1| \left(\frac{t}{1-t}\right)^{\frac{1}{2}}\right]^{\nu} e^{-\frac{\lambda^2}{2} t} dt$$

$$+ \int_0^{\frac{1}{(1+\bar{\beta}_1^2)} \cdot \frac{(V_1-\bar{\beta}_1)^2}{(1 + V_1^2)}} t^{-\frac{1}{2}} (1 - t)^{\frac{\nu-1}{2}}$$

$$\times \left[1 - |\bar{\beta}_1| \left(\frac{t}{1-t}\right)^{\frac{1}{2}}\right]^{\nu} e^{-\frac{\lambda^2}{2} t} \, dt \Bigg\}$$

$$\bar{\beta}_1 \leqq V_1 < \infty$$

For $\bar{\beta}_1 < 0$

(2.1.22 a-c)

$$H(V_1) \simeq \frac{\lambda}{2\sqrt{2\pi}}$$

$$\times \Bigg\{ \int_0^{\frac{1}{(1+\bar{\beta}_1^2)}} t^{-\frac{1}{2}} (1-t)^{\frac{\nu-1}{2}}$$

$$\times \left[1 - |\bar{\beta}_1| \left(\frac{t}{1-t}\right)^{\frac{1}{2}}\right]^{\nu} e^{-\frac{\lambda^2}{2} t} \, dt$$

$$- \int_0^{\frac{1}{(1+\bar{\beta}_1^2)} \cdot \frac{(V_1-\bar{\beta}_1)^2}{(1+V_1^2)}} t^{-\frac{1}{2}} (1-t)^{\frac{\nu-1}{2}}$$

$$\times \left[1 - |\bar{\beta}_1| \left(\frac{t}{1-t}\right)^{\frac{1}{2}}\right]^{\nu} e^{-\frac{\lambda^2}{2} t} \, dt \Bigg\}$$

$$-\infty < V_1 \leqq \bar{\beta}_1$$

$$H(V_1) \simeq \frac{\lambda}{2\sqrt{2\pi}}$$

$$\times \Bigg\{ \int_0^{\frac{1}{(1+\bar{\beta}_1^2)}} t^{-\frac{1}{2}} (1-t)^{\frac{\nu-1}{2}}$$

$$\times \left[1 - |\bar{\beta}_1| \; (\tfrac{t}{1-t})^{\frac{1}{2}\nu}\right] \; e^{-\frac{\lambda^2}{2} t} \; dt$$

$$+ \int_0^{\frac{1}{(1+\bar{\beta}_1{}^2)} \cdot \frac{(V_1 - \bar{\beta}_1)^2}{(1 + V_1{}^2)}} t^{-\frac{1}{2}} \; (1 - t)^{\frac{\nu-1}{2}}$$

$$\times \left[1 + |\bar{\beta}_1| \; (\tfrac{t}{1-t})^{\frac{1}{2}\nu}\right] \; e^{-\frac{\lambda^2}{2} t} \; dt \Big\}$$

$$\bar{\beta}_1 \leqq V_1 \leqq -\frac{1}{\bar{\beta}_1}$$

$$H(V_1) \simeq \frac{\lambda}{2\sqrt{2\pi}}$$

$$\times \Big\{ \int_0^{\frac{1}{(1+\bar{\beta}_1{}^2)}} t^{-\frac{1}{2}} \; (1 - t)^{\frac{\nu-1}{2}}$$

$$\times \left[1 - |\bar{\beta}_1| \; (\tfrac{t}{1-t})^{\frac{1}{2}\nu}\right] \; e^{-\frac{\lambda^2}{2} t} \; dt$$

$$+ \int_0^1 t^{-\frac{1}{2}} \; (1 - t)^{\frac{\nu-1}{2}}$$

$$\times \left[1 + |\bar{\beta}_1| \; (\tfrac{t}{1-t})^{\frac{1}{2}\nu}\right] \; e^{-\frac{\lambda^2}{2} t} \; dt$$

$$+ \int_0^1 t^{-\frac{1}{2}} \; (1 - t)^{\frac{\nu-1}{2}} \; (-)^{\nu}$$

$$\times \left[1 - |\bar{\beta}_1| \ (\tfrac{t}{1-t})^{\frac{1}{2}\nu}\right] e^{-\frac{\lambda^2}{2} t} \ dt$$

$$- \int_0^{\frac{1}{(1+\bar{\beta}_1^{\,2})} \cdot \frac{(V_1-\bar{\beta}_1)^2}{(1+V_1^{\,2})}} t^{-\frac{1}{2}} (1-t)^{\frac{\nu-1}{2}} (-)^{\nu}$$

$$\times \left[1 - | \bar{\beta}_1 | \ (\tfrac{t}{1-t})^{\frac{1}{2}\nu}\right] e^{-\frac{\lambda^2}{2} t} \ dt \Big\}$$

$$- \frac{1}{\bar{\beta}_1} \leqq V_1 < \infty$$

By use of the series expansion for $e^{-\frac{\lambda^2}{2} t}$ provided by (A.4) and the binomial formula we can write (2.1.20), (2.1.21), and (2.1.22) in terms of incomplete beta functions as follows:

For $\bar{\beta}_1 = 0$

(2.1.23 a-b) $$H(V_1) \simeq \frac{\bar{\mu}}{2\sqrt{2\pi}} \sum_{p=0}^{\infty} \frac{\left(-\frac{\bar{\mu}^2}{2}\right)^p}{p!}$$

$$\times \left\{ I_B \left[\tfrac{1}{2} + p, \tfrac{\nu+1}{2}; 1\right] \right.$$

$$\left. - I_B \left[\tfrac{1}{2} + p, \tfrac{\nu+1}{2}; \frac{V_1^{\,2}}{1+V_1^{\,2}}\right] \right\}$$

$$- \infty < V_1 \leqq 0$$

$$H(V_1) \simeq \frac{\bar{\mu}}{2\sqrt{2\pi}} \sum_{p=0}^{\infty} \frac{\left(-\frac{\bar{\mu}^2}{2}\right)^p}{p!}$$

$$\times \left\{ I_B \left[\frac{1}{2} + p, \frac{\nu+1}{2} ; 1\right] \right.$$

$$\left. + I_B \left[\frac{1}{2} + p, \frac{\nu+1}{2} ; \frac{V_1^2}{1+V_1^2}\right] \right\}$$

$$0 \leqq V_1 < \infty$$

For $\bar{\beta}_1 > 0$

(2.1.24 a-c)
$$H(V_1) \simeq \frac{\lambda}{2\sqrt{2\pi}} \sum_{p=0}^{\infty} \frac{\left(-\frac{\lambda^2}{2}\right)^p}{p!}$$

$$\times \sum_{k=0}^{\nu} \binom{\nu}{k} |\bar{\beta}_1|^k (-)^{\nu-k}$$

$$\times \left\{ I_B \left[\frac{k+1}{2} + p, \frac{\nu-k+1}{2} ; \frac{1}{(1+\bar{\beta}_1^2)} \cdot \frac{(V_1-\bar{\beta}_1)^2}{(1+V_1^2)} \right] \right.$$

$$\left. - I_B \left[\frac{k+1}{2} + p, \frac{\nu-k+1}{2} ; \frac{1}{(1+\bar{\beta}_1^2)} \right] \right\}$$

$$-\infty < V_1 \leqq -\frac{1}{\bar{\beta}_1}$$

$$H(V_1) \simeq \frac{\lambda}{2\sqrt{2\pi}} \cdot \sum_{p=0}^{\infty} \frac{\left(-\frac{\lambda^2}{2}\right)^p}{p!}$$

$$\times \sum_{k=0}^{\nu} \binom{\nu}{k} |\bar{\beta}_1|^k$$

$$\times \left\{ (-)^{\nu-k} \; I_B\left[\frac{k+1}{2} + p, \; \frac{\nu-k+1}{2} \; ; \; 1\right]\right.$$

$$- (-)^{\nu-k} \; I_B\left[\frac{k+1}{2} + p, \; \frac{\nu-k+1}{2}; \; \frac{1}{(1+\bar{\beta}_1^{\,2})}\right]$$

$$+ I_B\left[\frac{k+1}{2} + p, \; \frac{\nu-k+1}{2} \; ; \; 1\right]$$

$$\left. - I_B\left[\frac{k+1}{2} + p, \; \frac{\nu-k+1}{2}; \; \frac{1}{(1+\bar{\beta}_1^{\,2})} \cdot \frac{(V_1-\bar{\beta}_1)^2}{(1+V_1^{\,2})}\right]\right\}$$

$$-\frac{1}{\bar{\beta}_1} \leqq V_1 \leqq \bar{\beta}_1$$

$$H(V_1) \simeq \frac{\lambda}{2\sqrt{2\pi}} \sum_{p=0}^{\infty} \frac{\left(-\frac{\lambda^2}{2}\right)^p}{p!}$$

$$\times \sum_{k=0}^{\nu} \binom{\nu}{k} |\bar{\beta}_1|^k \; .$$

$$\times \left\{ (-)^{\nu-k} \; I_B\left[\frac{k+1}{2} + p,\ \frac{\nu-k+1}{2};\ \ 1\right]\right.$$

$$- (-)^{\nu-k} \; I_B\left[\frac{k+1}{2} + p,\ \frac{\nu-k+1}{2};\ \frac{1}{(1+\bar{\beta}_1^{\,2})}\right]$$

$$+ I_B\left[\frac{k+1}{2} + p,\ \frac{\nu-k+1}{2};\ \ 1\right]$$

$$\left. + (-)^{k} \; I_B\left[\frac{k+1}{2} + p,\ \frac{\nu-k+1}{2}\ ;\ \frac{1}{(1+\bar{\beta}_1^{\,2})} \cdot \frac{(V_1-\bar{\beta}_1)^2}{(1+V_1^{\,2})}\right]\right\}$$

$$\bar{\beta}_1 \leqq V_1 < \infty$$

For $\bar{\beta}_1 < 0$

(2.1.25 a-c) $$H(V_1) \simeq \frac{\lambda}{2\sqrt{2\pi}} \sum_{p=0}^{\infty} \frac{\left(-\frac{\lambda^2}{2}\right)^p}{p!}$$

$$\times \sum_{k=0}^{\nu} \binom{\nu}{k} |\bar{\beta}_1|^k \, (-)^k$$

$$\times \left\{ I_B\left[\frac{k+1}{2} + p,\ \frac{\nu-k+1}{2}\ ;\ \frac{1}{(1+\bar{\beta}_1^{\,2})}\right]\right.$$

$$\left. - I_B\left[\frac{k+1}{2} + p,\ \frac{\nu-k+1}{2};\ \frac{1}{(1+\bar{\beta}_1^{\,2})} \cdot \frac{(V_1-\bar{\beta}_1)^2}{(1+V_1^{\,2})}\right]\right\}$$

$$-\infty < V_1 \leqq \bar{\beta}_1$$

$$H(V_1) \simeq \frac{\lambda}{2\sqrt{2\pi}} \sum_{p=0}^{\infty} \frac{\left(-\frac{\lambda^2}{2}\right)^p}{p!}$$

$$\times \sum_{k=0}^{\nu} \binom{\nu}{k} |\bar{\beta}_1|^k$$

$$\times \left\{(-)^k \, I_B\left[\frac{k+1}{2} + p, \frac{\nu-k+1}{2}; \frac{1}{(1+\bar{\beta}_1^{\,2})}\right]\right.$$

$$\left. + I_B\left[\frac{k+1}{2} + p, \frac{\nu-k+1}{2}; \frac{1}{(1+\bar{\beta}_1^{\,2})} \cdot \frac{(V_1-\bar{\beta}_1)^2}{(1+V_1^{\,2})}\right]\right\}$$

$$\bar{\beta}_1 \leqq V_1 \leqq -\frac{1}{\bar{\beta}_1}$$

$$H(V_1) \simeq \frac{\lambda}{2\sqrt{2\pi}} \sum_{p=0}^{\infty} \frac{\left(-\frac{\lambda^2}{2}\right)^p}{p!}$$

$$\times \sum_{k=0}^{\nu} \binom{\nu}{k} |\bar{\beta}_1|^k$$

$$\times \left\{(-)^k \, I_B\left[\frac{k+1}{2} + p, \frac{\nu-k+1}{2}; \frac{1}{(1+\bar{\beta}_1^{\,2})}\right]\right.$$

$$+ \; I_B\left[\frac{k+1}{2} + p, \; \frac{\nu-k+1}{2} \; ; \; 1\right]$$

$$+ \; (-)^{\nu-k} \; I_B\left[\frac{k+1}{2} + p, \; \frac{\nu-k+1}{2}; \; 1\right]$$

$$- \; (-)^{\nu-k} \; I_B\left[\frac{k+1}{2} + p, \; \frac{\nu-k+1}{2}; \; \frac{1}{(1+\bar{\beta}_1^{\,2})} \cdot \frac{(V_1-\bar{\beta}_1)^2}{(1+V_1^{\,2})}\right]\Bigg\}$$

$$- \frac{1}{\bar{\beta}_1} \leq V_1 < \infty$$

2.2 Some Computational Results

We can express the incomplete beta functions of (2.1.23), (2.1.24), and (2.1.25) in terms of hypergeometric functions by means of (A.36b) when the arguments of the incomplete beta functions are less than one and in the closed form specified by (A.37) when the arguments are equal to one.

A computer program in the Fortran IV language with double precision real variables has been written suitable to approximate $H(V_1)$ according to (2.1.23), (2.1.24), and (2.1.25). In the following tables we present some examples of approximations to $H(V_1)$ for $\nu = 0, 1, 2, 3, 4, 5$, $\bar{\beta}_1 = 0, 1$, and $\bar{\mu}^2 = 10$. We will take into account the approximation made in (2.1.13) by computing an appropriate normalizing constant; i.e. we approximate $H(V_1)$ according to (2.1.23), (2.1.24), and (2.1.25) subject to the restriction that $H(V_1) \to 1$ as $V_1 \to \infty$.

TABLE 2.2.1

Normalizing Constants: $\bar{\mu}^2 = 10$

ν	$\bar{\beta}_1 = 0$	$\bar{\beta}_1 = 1$
0	1.0700	1.0293
1	.9980	1.0000
2	.9448	1.0293
3	.9000	1.0998
4	.8630	1.2131
5	.8295	1.3694

TABLE 2.2.2

$H(V_1)$: $\bar{\mu}^2 = 10$, $\bar{\beta}_1 = 0$

V_1	$\nu=0$ $H(V_1)$	$\nu=1$ $H(V_1)$	$\nu=2$ $H(V_1)$	$\nu=3$ $H(V_1)$	$\nu=4$ $H(V_1)$	$\nu=5$ $H(V_1)$
-1.0	.0213	.0128	.0076	.0051	.0037	.0025
- .8	.0363	.0240	.0170	.0126	.0089	.0075
- .6	.0672	.0518	.0411	.0329	.0272	.0227
- .4	.1380	.1197	.1060	.0942	.0844	.0761
- .2	.2816	.2674	.2555	.2449	.2353	.2270
0.0	.5000	.5000	.5000	.5000	.5000	.4999
.2	.7189	.7329	.7437	.7558	.7643	.7734
.4	.8627	.8806	.8933	.9067	.9146	.9242
.6	.9335	.9484	.9580	.9680	.9718	.9786
.8	.9643	.9772	.9821	.9883	.9901	.9937
1.0	.9793	.9883	.9915	.9957	.9954	.9987
1.2	.9853	.9935	.9950	.9972	.9968	.9996
1.4	.9889	.9959	.9962	.9987	.9976	.9984
1.6	.9927	.9970	.9956	.9988	.9960	.9982
1.8	.9931	.9974	.9966	.9973	.9970	.9972
2.0	.9963	.9973	.9964	.9970	.9967	.9975

TABLE 2.2.3

$H(V_1)$: $\bar{\mu}^2 = 10$, $\bar{\beta}_1 = 1$

V_1	$\nu=0$ $H(V_1)$	$\nu=1$ $H(V_1)$	$\nu=2$ $H(V_1)$	$\nu=3$ $H(V_1)$	$\nu=4$ $H(V_1)$	$\nu=5$ $H(V_1)$
0.0	.0002	.0030	.0029	.0022	.0065	.0085
.2	.0094	.0125	.0172	.0214	.0274	.0334
.4	.0440	.0612	.0812	.1025	.1265	.1514
.6	.1462	.1933	.2422	.2926	.3437	.3939
.8	.3154	.3947	.4688	.5379	.6019	.6592
1.0	.4997	.5942	.6736	.7399	.7945	.8392
1.2	.6526	.7451	.8128	.8644	.9025	.9308
1.4	.7626	.8423	.8949	.9310	.9547	.9710
1.6	.8358	.9017	.9401	.9644	.9786	.9876
1.8	.8836	.9370	.9648	.9813	.9897	.9946
2.0	.9150	.9582	.9788	.9896	.9945	.9976
2.2	.9357	.9719	.9864	.9939	.9974	.9992
2.4	.9506	.9801	.9909	.9968	.9982	.9995
2.6	.9608	.9855	.9943	.9977	.9988	1.0002
2.8	.9681	.9890	.9958	.9985	.9997	1.0002
3.0	.9734	.9920	.9967	.9989	.9996	.9998

CHAPTER 3

THE DISTRIBUTION FUNCTION OF THE GCL IDENTIFIABILITY TEST STATISTIC

3.1 The Asymptotic Distribution Function

Richardson (1968a) has derived the exact finite sample density function of the identifiability test statistic defined by

$$(3.1.1) \quad F = \frac{Q(\hat{\beta}_{.1})/\nu}{G_2(\hat{\beta}_{.1})/m}$$

where m is defined by

$$(3.1.2) \quad m = N-K$$

From (1.2.3) and (1.2.5) we have equivalent expressions for F

$$(3.1.3\ \text{a-b}) \quad F = \frac{m}{N} \cdot \frac{\bar{\omega}_{11}}{\tilde{\omega}_{11}}$$

$$F = \frac{V_2/\nu}{V_3/m}$$

The density function of F is

$$(3.1.4\ \text{a-b}) \quad g(F) = \frac{(\nu/m)}{B(\nu/2,\ m/2)} \quad e^{-\frac{\bar{\mu}^2}{2}(1+\bar{\beta}_1^{\,2})}$$

$$\times \left[\frac{\Gamma\left(\frac{\nu+2}{2}\right)\Gamma\left(\frac{m+\nu+1}{2}\right)}{\Gamma\left(\frac{\nu+1}{2}\right)\Gamma\left(\frac{m+\nu+2}{2}\right)} \right] \sum_{j=0}^{\infty} \sum_{s=0}^{\infty} \sum_{\ell=0}^{\infty}$$

$$\frac{\left(\frac{\bar{\beta}_1^{2} \bar{\mu}^{-2}}{2}\right)^{s+\ell} \left(\frac{\bar{\mu}^{-2}}{2}\right)^{j} \left(\frac{\nu}{m} F\right)^{s+\frac{\nu}{2}-1}}{s!\ \ell!\ j!\ \left(1 + \frac{\nu}{m} F\right)^{\frac{m+\nu}{2}+s}}$$

$$\times \frac{\left(\frac{\nu+2}{2}\right)_{j+\ell} \left(\frac{1}{2}\right)_{j+\ell} \left(\frac{m+\nu+1}{2}\right)_{j} \left(\frac{m+\nu}{2}\right)_{s}}{\left(\frac{\nu+1}{2}\right)_{s+j+\ell} \left(\frac{1}{2}\right)_{j} \left(\frac{m+\nu+2}{2}\right)_{j+\ell}}$$

$$\times\ {}_2F_1 \left[\begin{matrix} \frac{1}{2} + \ell, & \frac{m+\nu}{2} + s; & \\ & & \frac{1}{1 + \frac{\nu}{m} F} \\ \frac{m+\nu+2}{2} + j + \ell; & & \end{matrix} \right]$$

$0 \leq F < \infty$

$= 0$ otherwise

(Richardson, 1968a, p. 209).

A random variable distributed as (central) F with ν and m degrees of freedom has the density function

$$\text{(3.1.5 a-b)} \quad \bar{g}(F) = \frac{\left(\frac{\nu}{m}\right)}{B(\nu/2, m/2)} \cdot \frac{\left(\frac{\nu}{m} F\right)^{\frac{\nu}{2} - 1}}{\left(1 + \frac{\nu}{m} F\right)^{\frac{m+\nu}{2}}}$$

$0 \leqq F < \infty$

$= 0$ otherwise.

Let the distribution function of F in (3.1.5) be defined by

(3.1.6 a-b) $$\bar{G}(F) = \int_{-\infty}^{F} \bar{g}(f)\, df$$

$0 \leqq F < \infty$

$= 0$ otherwise.

From (3.1.5), (3.1.6), (A.34), and (A.36) we obtain

(3.1.7 a-b) $$\bar{G}(F) = \frac{\left(\frac{\nu}{m} F\right)^{\frac{\nu}{2}}}{(\nu/2)B(\nu/2,m/2)\left(1 + \frac{\nu}{m} F\right)^{\frac{m+\nu}{2}}} \times {}_2F_1\left[\begin{matrix} 1, \frac{m+\nu}{2}; \\ \frac{\nu+2}{2}; \end{matrix} \; \frac{\frac{\nu}{m} F}{1 + \frac{\nu}{m} F}\right]$$

$0 \leqq F < \infty$

$= 0$ otherwise.

Let the distribution function of the identifiability test statistic be defined by

$$(3.1.8\ \text{a-b})\quad G(F) = \int_{-\infty}^{F} g(f)\, df$$

$$0 \leqq F < \infty$$

$$= 0 \quad \text{otherwise.}$$

From (3.1.4), (3.1.8), (A.2), (A.4), (A.34), and (A.36) we obtain

$$(3.1.9\ \text{a-b})\quad G(F) = \frac{\left(\frac{\nu}{m} F\right)^{\frac{\nu}{2}}}{(\nu/2)B(\nu/2,m/2)(1 + \frac{\nu}{m} F)^{\frac{m+\nu}{2}}}$$

$$\times\ e^{-\frac{\bar{\mu}^2}{2}(1+\bar{\beta}_1^{\,2})} \left[\frac{\Gamma\left(\frac{\nu+2}{2}\right)\Gamma\left(\frac{m+\nu+1}{2}\right)}{\Gamma\left(\frac{\nu+1}{2}\right)\Gamma\left(\frac{m+\nu+2}{2}\right)} \right]$$

$$\times \sum_{s,j,\ell,t,r=0}^{\infty} \frac{\left(\frac{\nu}{m} F\right)^{s+r}}{(1 + \frac{\nu}{m} F)^{s+r+t}} \cdot \frac{\left(\frac{\bar{\mu}^2}{2}\right)^{j} \left(\frac{\bar{\beta}_1^{\,2}\bar{\mu}^2}{2}\right)^{s+\ell}}{s!\ \ j!\ \ \ell!\ \ t!}$$

$$\times \frac{\left(\frac{\nu+2}{2}\right)_{j+\ell}\left(\frac{1}{2}\right)_{j+\ell}\left(\frac{m+\nu+1}{2}\right)_{j}}{\left(\frac{\nu+1}{2}\right)_{s+j+\ell}\left(\frac{1}{2}\right)_{j}}$$

$$\times \frac{\left(\frac{m+\nu}{2}\right)_{s+t+r}\left(\frac{\nu}{2}\right)_{s}\left(\frac{1}{2} + \ell\right)_{t}}{\left(\frac{m+\nu+2}{2}\right)_{j+\ell+t}\left(\frac{\nu+2}{2}\right)_{s+r}}$$

$0 \leqq F < \infty$

$= 0$ otherwise.

Using (A.2) and (A.4) we write (3.1.9) as

(3.1.10 a-b) $$G(F) = \frac{\left(\frac{\nu}{m} F\right)^{\frac{\nu}{2}}}{(\nu/2)B(\nu/2,m/2)(1+\frac{\nu}{m}F)^{\frac{m+\nu}{2}}}$$

$$\times\ e^{-\frac{\bar{\mu}^2}{2}(1+\bar{\beta}_1^{\,2})} \left[\frac{\Gamma\left(\frac{\nu+2}{2}\right)\Gamma\left(\frac{m+\nu+1}{2}\right)}{\Gamma\left(\frac{\nu+1}{2}\right)\Gamma\left(\frac{m+\nu+2}{2}\right)}\right]$$

$$\times \sum_{s,j,t,r=0}^{\infty} \frac{\left(\frac{\nu}{m}F\right)^{s+r}}{(1+\frac{\nu}{m}F)^{s+r+t}} \cdot \frac{\left(\frac{\bar{\mu}^2}{2}\right)^j \left(\frac{\bar{\beta}_1^{\,2}\bar{\mu}^2}{2}\right)^s}{s!\quad j!\quad t!}$$

$$\times \frac{\left(\frac{\nu+2}{2}\right)_j \left(\frac{m+\nu+1}{2}\right)_j \left(\frac{m+\nu}{2}\right)_{s+t+r} \left(\frac{\nu}{2}\right)_s \left(\frac{1}{2}\right)_t}{\left(\frac{\nu+1}{2}\right)_{s+j}\left(\frac{m+\nu+2}{2}\right)_{j+t}\left(\frac{\nu+2}{2}\right)_{s+r}}$$

$$\times\ {}_3F_3\left[\begin{matrix}\frac{\nu+2}{2}+j,\ \frac{1}{2}+j,\ \frac{1}{2}+t; \\ \frac{\nu+1}{2}+s+j,\ \frac{m+\nu+2}{2}+j+t,\ \frac{1}{2};\end{matrix}\ \frac{\bar{\beta}_1^{\,2}\bar{\mu}^2}{2}\right]$$

$0 \leqq F < \infty$

$= 0$ otherwise.

By repeated application of (A.11) and by use of (A.13) we have

$$(3.1.11)\qquad {}_3F_3\left[\begin{matrix}\frac{\nu+2}{2}+j,\ \frac{1}{2}+j,\ \frac{1}{2}+t;\\ \frac{\nu+1}{2}+s+j,\ \frac{m+\nu+2}{2}+j+t,\ \frac{1}{2};\end{matrix}\ \frac{\bar{\beta}_1^2\mu^{-2}}{2}\right]$$

$$= e^{\frac{\bar{\beta}_1^2\mu^{-2}}{2}} \sum_{\ell=0}^{\infty} \frac{\left(\frac{\bar{\beta}_1^2\mu^{-2}}{2}\right)^{\ell}}{\ell!} \cdot \frac{\left(-\frac{1}{2}+s\right)_\ell\left(\frac{\nu}{2}+s\right)_\ell\left(\frac{1}{2}+t\right)_\ell}{\left(\frac{\nu+1}{2}+s+j\right)_\ell\left(\frac{m+\nu+2}{2}+j+t\right)_\ell\left(\frac{1}{2}\right)_\ell}$$

$$\times \sum_{p=0}^{\infty} \frac{\left(\frac{\bar{\beta}_1^2\mu^{-2}}{2}\right)^{p}}{p!} \cdot \frac{\left(\frac{m+\nu}{2}+s+t+\ell\right)_p\left(\frac{m+\nu+1}{2}+j\right)_p}{\left(\frac{m+\nu+2}{2}+j+t+\ell\right)_p\left(\frac{1}{2}+\ell\right)_p}$$

$$\times \sum_{q=0}^{\infty} \frac{\left(-\frac{\bar{\beta}_1^2\mu^{-2}}{2}\right)^{q}}{q!} \cdot \frac{\left(\frac{m+\nu}{2}+s+\ell+p\right)_q}{\left(\frac{1}{2}+\ell+p\right)_q}$$

Replacing $\underline{q}$ by $\underline{q-p}$ in (3.1.11) and making use of (A.2), (A.4), and (A.7) we obtain

$$(3.1.12)\qquad {}_3F_3\left[\begin{matrix}\frac{\nu+2}{2}+j,\ \frac{1}{2}+j,\ \frac{1}{2}+t;\\ \frac{\nu+1}{2}+s+j,\ \frac{m+\nu+2}{2}+j+t,\ \frac{1}{2};\end{matrix}\ \frac{\bar{\beta}_1^2\mu^{-2}}{2}\right]$$

$$= e^{\frac{\bar{\beta}_1^2 \bar{\mu}^{-2}}{2}} \sum_{\ell=0}^{\infty} \frac{\left(\frac{\bar{\beta}_1^2 \bar{\mu}^{-2}}{2}\right)^{\ell}}{\ell!} \cdot \frac{\left(-\frac{1}{2}+s\right)_{\ell}\left(\frac{\nu}{2}+s\right)_{\ell}\left(\frac{1}{2}+t\right)_{\ell}}{\left(\frac{\nu+1}{2}+s+j\right)_{\ell}\left(\frac{m+\nu+2}{2}+j+t\right)_{\ell}\left(\frac{1}{2}\right)_{\ell}}$$

$$\times \sum_{q=0}^{\infty} \frac{\left(-\frac{\bar{\beta}_1^2 \bar{\mu}^{-2}}{2}\right)^{q}}{q!} \cdot \frac{\left(\frac{m+\nu}{2}+s+\ell\right)_q}{\left(\frac{1}{2}+\ell\right)_q}$$

$$\times \; {}_3F_2\left[\begin{matrix} -q, \frac{m+\nu}{2}+s+t+\ell, \quad \frac{m+\nu+1}{2}+j; \\ \frac{m+\nu+2}{2}+j+t+\ell, \quad \frac{m+\nu}{2}+s+\ell; \end{matrix} \; 1\right]$$

We have

$$(3.1.13) \quad {}_3F_2\left[\begin{matrix} -q, \frac{m+\nu}{2}+s+t+\ell, \frac{m+\nu+1}{2}+j; \\ \frac{m+\nu+2}{2}+j+t+\ell, \quad \frac{m+\nu}{2}+s+\ell; \end{matrix} \; 1\right]$$

$$= \frac{\left(\frac{1}{2}\right)_{\ell+q}\left(\frac{m+\nu+2}{2}\right)_{j+t+\ell}}{\left(\frac{1}{2}\right)_{\ell+t}\left(\frac{m+\nu+2}{2}\right)_{j+q+\ell}}$$

$$\times \; {}_3F_2\left[\begin{matrix} -t, \frac{m+\nu}{2}+s+\ell+q, \frac{m+\nu+1}{2}+j; \\ \frac{m+\nu+2}{2}+j+\ell+q, \quad \frac{m+\nu}{2}+s+\ell; \end{matrix} \; 1\right]$$

by (A.2) and (Bailey, 1964, pp. 16-19).

From (3.1.10), (3.1.12), (3.1.13), (A.2), and (A.4) we obtain

$$
\text{(3.1.14a-b)}\quad G(F) = \frac{\left(\frac{\nu}{m}F\right)^{\frac{\nu}{2}}}{(\nu/2)B(\nu/2,m/2)\left(1+\frac{\nu}{m}F\right)^{\frac{m+\nu}{2}}}
$$

$$
\times\ e^{-\frac{\bar{\mu}^2}{2}} \left[\frac{\Gamma\left(\frac{\nu+2}{2}\right)\Gamma\left(\frac{m+\nu+1}{2}\right)}{\Gamma\left(\frac{\nu+1}{2}\right)\Gamma\left(\frac{m+\nu+2}{2}\right)}\right]
$$

$$
\times \sum_{r,s,t,\ell,q,p=0}^{\infty} \frac{\left(\frac{\nu}{m}F\right)^{s+r}}{\left(1+\frac{\nu}{m}F\right)^{s+r+t}} \cdot \frac{(-)^q\left(\frac{\bar{\beta}_1^2\bar{\mu}^2}{2}\right)^{s+\ell+q}}{s!\ t!\ \ell!\ q!\ p!}
$$

$$
\times \frac{\left(\frac{m+\nu+1}{2}\right)_p\left(\frac{m+\nu}{2}\right)_{s+t+r}\left(\frac{\nu}{2}\right)_{s+\ell}(-t)_p}{\left(\frac{\nu+1}{2}\right)_{s+\ell}\left(\frac{\nu+2}{2}\right)_{s+r}}
$$

$$
\times \frac{\left(\frac{m+\nu}{2}+s+\ell+p\right)_q\left(-\frac{1}{2}+s\right)_\ell}{\left(\frac{m+\nu+2}{2}\right)_{q+\ell+p}}
$$

$$
\times\ {}_2F_2\left[\begin{matrix}\frac{\nu+2}{2},\ \frac{m+\nu+1}{2}+p; & \\ & \frac{\bar{\mu}^2}{2} \\ \frac{\nu+1}{2}+s+\ell,\ \frac{m+\nu+2}{2}+q+\ell+p; & \end{matrix}\right]
$$

$$0 \leqq F < \infty$$

$$= 0 \quad \text{otherwise}.$$

By (A.2) and (A.18)

$$\text{(3.1.15 a-b)} \quad \lim_{\bar{\mu}^2 \to \infty} G(F) = \lim_{\bar{\mu}^2 \to \infty} \frac{\left(\frac{\nu}{m} F\right)^{\frac{\nu}{2}}}{(\nu/2)B(\nu/2,m/2)\left(1 + \frac{\nu}{m} F\right)^{\frac{m+\nu}{2}}}$$

$$\times\, e^{-\frac{\bar{\mu}^2}{2}} \left[\frac{\Gamma\left(\frac{\nu+2}{2}\right)\Gamma\left(\frac{m+\nu+1}{2}\right)}{\Gamma\left(\frac{\nu+1}{2}\right)\Gamma\left(\frac{m+\nu+2}{2}\right)}\right]$$

$$\times \sum_{s,t,r,p=0}^{\infty} \frac{\left(\frac{\nu}{m} F\right)^{s+r}}{\left(1 + \frac{\nu}{m} F\right)^{s+r+t}} \cdot \frac{\left(\frac{\bar{\beta}_1^2 \bar{\mu}^2}{2}\right)^s}{s!\; t!\; p!}$$

$$\times \frac{\left(\frac{m+\nu+1}{2}\right)_p \left(\frac{m+\nu}{2}\right)_{s+t+r} \left(\frac{\nu}{2}\right)_s (-t)_p}{\left(\frac{\nu+1}{2}\right)_s \left(\frac{\nu+2}{2}\right)_{s+r} \left(\frac{m+\nu+2}{2}\right)_p}$$

$$\times \sum_{q=0}^{\infty} \frac{\left(\frac{m+\nu}{2} + s + p\right)_q}{\left(\frac{m+\nu+2}{2} + p\right)_q} \cdot \frac{\left(-\frac{\bar{\beta}_1^2 \bar{\mu}^2}{2}\right)^q}{q!}$$

$$\times \; {}_2F_2 \left[\begin{matrix} \frac{\nu+2}{2}, & \frac{m+\nu+1}{2} + p; \\ \frac{\nu+1}{2} + s, & \frac{m+\nu+2}{2} + q + p; \end{matrix} \; \frac{\bar{\mu}^2}{2} \right]$$

$$0 \leqq F < \infty$$

$$= 0 \quad \text{otherwise.}$$

Making use of (A.2), (A.4), (A.18), and (A.33) we have

$$\text{(3.1.16 a-b)} \quad G(F) \to \frac{\left(\frac{\nu}{m} F\right)^{\frac{\nu}{2}} \left(1 + \bar{\beta}_1^{\,2}\right)^{-\frac{m+\nu}{2}}}{(\nu/2) B(\nu/2, m/2) \left(1 + \frac{\nu}{m} F\right)^{\frac{m+\nu}{2}}}$$

$$\times \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \sum_{r=0}^{\infty} \frac{\left(\frac{\nu}{m} F\right)^{s+r}}{\left(1 + \frac{\nu}{m} F\right)^{s+r+t}} \cdot \frac{\left(\frac{\bar{\beta}_1^{\,2}}{1+\bar{\beta}_1^{\,2}}\right)^s}{s! \quad t!}$$

$$\times \frac{\left(\frac{m+\nu}{2}\right)_{s+t+r} \left(\frac{\nu}{2}\right)_s}{\left(\frac{\nu+2}{2}\right)_{s+r}} \sum_{p=0}^{\infty} \frac{(-t)_p \left(1 + \bar{\beta}_1^{\,2}\right)^{-p}}{p!}$$

$$0 \leqq F < \infty$$

$$= 0 \quad \text{otherwise}$$

as $\bar{\mu}^2 \to \infty$.

By use of (A.2), (A.4), (A.6), and (A.11), (3.1.16) can be written as follows:

$$\text{(3.1.17 a-b)}\quad G(F) \to \frac{\left(\dfrac{\frac{\nu}{m}F}{1+\frac{\nu}{m}F}\right)^{\frac{\nu}{2}} \left(1+\beta_1^{\,2}\right)^{-\frac{m+\nu}{2}}}{(\nu/2)\ B(\nu/2,m/2)}$$

$$\times \sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\sum_{r=0}^{\infty} \frac{\left(\dfrac{\frac{\nu}{m}F}{1+\frac{\nu}{m}F}\right)^{s+r}\left(\dfrac{\bar{\beta}_1^{\,2}}{1+\bar{\beta}_1^{\,2}}\right)^{s+t}}{s!\qquad r!\qquad t!}$$

$$\times \frac{\left(\frac{m+\nu}{2}\right)_{s+t}\left(\frac{\nu}{2}\right)_{s+r}\left(1-\frac{m}{2}-t\right)_r}{\left(\frac{\nu+2}{2}\right)_{s+r}}$$

$$0 \leqq F < \infty$$

$$= 0 \quad \text{otherwise}$$

as $\bar{\mu}^2 \to \infty$.

We have

$$\text{(3.1.18)}\quad \sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\sum_{r=0}^{\infty} \frac{\left(\dfrac{\frac{\nu}{m}F}{1+\frac{\nu}{m}F}\right)^{s+r}\left(\dfrac{\bar{\beta}_1^{\,2}}{1+\bar{\beta}_1^{\,2}}\right)^{s+t}}{s!\qquad r!\qquad t!}$$

$$\times \frac{\left(\frac{m+\nu}{2}\right)_{s+t}\left(\frac{\nu}{2}\right)_{s+r}\left(1-\frac{m}{2}-t\right)_r}{\left(\frac{\nu+2}{2}\right)_{s+r}}$$

$$= \sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\sum_{r=0}^{t} \frac{\left(\frac{\frac{\nu}{m}F}{1+\frac{\nu}{m}F}\right)^{s+r}\left(\frac{\bar{\beta}_1^{\,2}}{1+\bar{\beta}_1^{\,2}}\right)^{s+t}}{s!\ r!\ t!}$$

$$\times \frac{\left(\frac{m+\nu}{2}\right)_{s+t}\left(\frac{\nu}{2}\right)_{s+r}\left(1-\frac{m}{2}-t\right)_r}{\left(\frac{\nu+2}{2}\right)_{s+r}}$$

$$+ \sum_{s=0}^{\infty}\sum_{r=1}^{\infty}\sum_{t=0}^{r-1} \frac{\left(\frac{\frac{\nu}{m}F}{1+\frac{\nu}{m}F}\right)^{s+r}\left(\frac{\bar{\beta}_1^{\,2}}{1+\bar{\beta}_1^{\,2}}\right)^{s+t}}{s!\ r!\ t!}$$

$$\times \frac{\left(\frac{m+\nu}{2}\right)_{s+t}\left(\frac{\nu}{2}\right)_{s+r}\left(1-\frac{m}{2}-t\right)_r}{\left(\frac{\nu+2}{2}\right)_{s+r}}$$

Replacing $\underline{t}$ by $\underline{t+r}$ in the first expression to the right of the equal sign in (3.1.18) and making use of (A.2) and (A.7) we obtain

$$(3.1.19) \quad \sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\sum_{r=0}^{t} \frac{\left(\frac{\frac{\nu}{m}F}{1+\frac{\nu}{m}F}\right)^{s+r}\left(\frac{\bar{\beta}_1^{\,2}}{1+\bar{\beta}_1^{\,2}}\right)^{s+t}}{s!\ r!\ t!}$$

$$\times \frac{\left(\frac{m+\nu}{2}\right)_{s+t}\left(\frac{\nu}{2}\right)_{s+r}\left(1-\frac{m}{2}-t\right)_r}{\left(\frac{\nu+2}{2}\right)_{s+r}} =$$

$$\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\sum_{r=0}^{\infty} \frac{(-)^r\left(\frac{\frac{\nu}{m}F}{1+\frac{\nu}{m}F}\right)^{s+r}\left(\frac{\bar{\beta}_1^{\,2}}{1+\bar{\beta}_1^{\,2}}\right)^{s+r+t}}{s!\qquad r!}$$

$$\times \frac{\left(\frac{m+\nu}{2}\right)_{s+t+r}\left(\frac{\nu}{2}\right)_{s+r}\left(\frac{m}{2}\right)_{t+r}}{\left(\frac{\nu+2}{2}\right)_{s+r}\left(\frac{m}{2}\right)_t\ (1)_{t+r}}$$

Replacing r by r-s on the right side of (3.1.19) and making use of (A.7) we obtain

$$(3.1.20)\quad \sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\sum_{r=0}^{t} \frac{\left(\frac{\frac{\nu}{m}F}{1+\frac{\nu}{m}F}\right)^{s+r}\left(\frac{\bar{\beta}_1^{\,2}}{1+\bar{\beta}_1^{\,2}}\right)^{s+t}}{s!\qquad r!\qquad t!}$$

$$\times \frac{\left(\frac{m+\nu}{2}\right)_{s+t}\left(\frac{\nu}{2}\right)_{s+r}\left(1-\frac{m}{2}-t\right)_r}{\left(\frac{\nu+2}{2}\right)_{s+r}} =$$

$$\sum_{t=0}^{\infty}\sum_{r=0}^{\infty} \frac{\left(\frac{-\frac{\nu}{m}F}{1+\frac{\nu}{m}F}\right)^{r}\left(\frac{\bar{\beta}_1^{\,2}}{1+\bar{\beta}_1^{\,2}}\right)^{r+t}}{r!}$$

$$\times \frac{\left(\frac{m+\nu}{2}\right)_{r+t}\left(\frac{\nu}{2}\right)_r\left(\frac{m}{2}\right)_{t+r}}{\left(\frac{\nu+2}{2}\right)_r\left(\frac{m}{2}\right)_t(1)_{t+r}}$$

$$\times \sum_{s=0}^{\infty} \frac{(-r)_s(-t-r)_s}{\left(1 - \frac{m}{2} - t - r\right)_s \quad s!}$$

Using (A.7) and (A.8) and replacing <u>t</u> by <u>t-r</u> on the right side of (3.1.20) we obtain

$$(3.1.21) \quad \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \sum_{r=0}^{t} \frac{\left(\frac{\frac{\nu}{m}F}{1+\frac{\nu}{m}F}\right)^{s+r} \left(\frac{\bar{\beta}_1^2}{1+\bar{\beta}_1^2}\right)^{s+t}}{s! \qquad r! \qquad t!}$$

$$\times \frac{\left(\frac{m+\nu}{2}\right)_{s+t}\left(\frac{\nu}{2}\right)_{s+r}\left(1 - \frac{m}{2} - t\right)_r}{\left(\frac{\nu+2}{2}\right)_{s+r}} =$$

$$\sum_{r=0}^{\infty} \frac{\left(\frac{\nu}{2}\right)_r\left(1 - \frac{m}{2}\right)_r}{\left(\frac{\nu+2}{2}\right)_r} \cdot \frac{\left(\frac{\frac{\nu}{m}F}{1+\frac{\nu}{m}F}\right)^r}{r!}$$

$$\times \sum_{t=r}^{\infty} \frac{\left(\frac{m+\nu}{2}\right)_t \left(\frac{\bar{\beta}_1^2}{1+\bar{\beta}_1^2}\right)^t}{t!}$$

Replacing r by r+t in the second expression to the right of the equal sign in (3.1.18) and making use of (A.2) and (A.7) we obtain

$$(3.1.22)\quad \sum_{s=0}^{\infty} \sum_{r=1}^{\infty} \sum_{t=0}^{r-1} \frac{\left(\frac{\frac{\nu}{m}F}{1+\frac{\nu}{m}F}\right)^{s+r} \left(\frac{\bar{\beta}_1^{\,2}}{1+\bar{\beta}_1^{\,2}}\right)^{s+t}}{s!\quad r!\quad t!}$$

$$\times \frac{\left(\frac{m+\nu}{2}\right)_{s+t}\left(\frac{\nu}{2}\right)_{s+r}\left(1-\frac{m}{2}-t\right)_r}{\left(\frac{\nu+2}{2}\right)_{s+r}} =$$

$$\sum_{s=0}^{\infty} \sum_{r=1}^{\infty} \sum_{t=0}^{\infty} \frac{(-)^t\left(\frac{\frac{\nu}{m}F}{1+\frac{\nu}{m}F}\right)^{s+r+t} \left(\frac{\bar{\beta}_1^{\,2}}{1+\bar{\beta}_1^{\,2}}\right)^{s+t}}{s!\quad t!}$$

$$\times \frac{\left(\frac{m+\nu}{2}\right)_{s+t}\left(\frac{\nu}{2}\right)_{s+r+t}\left(\frac{m}{2}\right)_t\left(1-\frac{m}{2}\right)_r}{\left(\frac{\nu+2}{2}\right)_{s+r+t}\quad (1)_{r+t}}$$

Replacing s by s-t on the right side of (3.1.22) and making use of (A.2) and (A.7) we obtain

$$(3.1.23)\quad \sum_{s=0}^{\infty} \sum_{r=1}^{\infty} \sum_{t=0}^{r-1} \frac{\left(\frac{\frac{\nu}{m}F}{1+\frac{\nu}{m}F}\right)^{s+r} \left(\frac{\bar{\beta}_1^{\,2}}{1+\bar{\beta}_1^{\,2}}\right)^{s+t}}{s!\quad r!\quad t!}$$

$$\times \frac{\left(\frac{m+\nu}{2}\right)_{s+t}\left(\frac{\nu}{2}\right)_{s+r}\left(1 - \frac{m}{2} - t\right)_r}{\left(\frac{\nu+2}{2}\right)_{s+r}} =$$

$$\sum_{r=1}^{\infty} \sum_{s=0}^{\infty} \frac{\left(\frac{\frac{\nu}{m} F}{1 + \frac{\nu}{m} F}\right)^{s+r} \left(\frac{\bar{\beta}_1^{\,2}}{1 + \bar{\beta}_1^{\,2}}\right)^{s}}{s! \quad r!}$$

$$\times \frac{\left(\frac{m+\nu}{2}\right)_{s}\left(\frac{\nu}{2}\right)_{s+r}\left(1 - \frac{m}{2}\right)_r}{\left(\frac{\nu+2}{2}\right)_{s+r}}$$

$$\times \sum_{t=0}^{\infty} \frac{(-s)_t \left(\frac{m}{2}\right)_t}{(1+r)_t \; t!}$$

Using (A.2) and (A.8) and replacing r by r-s on the right side of (3.1.23) we obtain

$$(3.1.24) \quad \sum_{s=0}^{\infty} \sum_{r=1}^{\infty} \sum_{t=0}^{r-1} \frac{\left(\frac{\frac{\nu}{m} F}{1 + \frac{\nu}{m} F}\right)^{s+r} \left(\frac{\bar{\beta}_1^{\,2}}{1 + \bar{\beta}_1^{\,2}}\right)^{s+t}}{s! \quad r! \quad t!}$$

$$\times \frac{\left(\frac{m+\nu}{2}\right)_{s+t}\left(\frac{\nu}{2}\right)_{s+r}\left(1 - \frac{m}{2} - t\right)_r}{\left(\frac{\nu+2}{2}\right)_{s+r}} =$$

$$\sum_{r=1}^{\infty} \frac{\left(\frac{\nu}{2}\right)_r \left(1 - \frac{m}{2}\right)_r}{\left(\frac{\nu+2}{2}\right)_r} \cdot \frac{\left(\frac{\frac{\nu}{m} F}{1 + \frac{\nu}{m} F}\right)^r}{r!}$$

$$\times \sum_{s=0}^{r-1} \frac{\left(\frac{m+\nu}{2}\right)_s \left(\frac{\bar{\beta}_1^{\,2}}{1 + \bar{\beta}_1^{\,2}}\right)^s}{s!}$$

From (3.1.18), (3.1.21), and (3.1.24) we have

$$(3.1.25) \quad \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \sum_{r=0}^{\infty} \frac{\left(\frac{\frac{\nu}{m} F}{1 + \frac{\nu}{m} F}\right)^{s+r} \left(\frac{\bar{\beta}_1^{\,2}}{1 + \bar{\beta}_1^{\,2}}\right)^{s+t}}{s! \quad r! \quad t!}$$

$$\times \frac{\left(\frac{m+\nu}{2}\right)_{s+t} \left(\frac{\nu}{2}\right)_{s+r} \left(1 - \frac{m}{2} - t\right)_r}{\left(\frac{\nu+2}{2}\right)_{s+r}} =$$

$$\sum_{r=0}^{\infty} \frac{\left(\frac{\nu}{2}\right)_r \left(1 - \frac{m}{2}\right)_r}{\left(\frac{\nu+2}{2}\right)_r} \cdot \frac{\left(\frac{\frac{\nu}{m} F}{1 + \frac{\nu}{m} F}\right)^r}{r!}$$

$$\times \sum_{t=0}^{\infty} \frac{\left(\frac{m+\nu}{2}\right)_t \left(\frac{\bar{\beta}_1^{\,2}}{1 + \bar{\beta}_1^{\,2}}\right)^t}{t!}$$

By use of (A.6) and (A.11), (3.1.25) can be written as

$$(3.1.26) \quad \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \sum_{r=0}^{\infty} \frac{\left(\dfrac{\frac{\nu}{m} F}{1 + \frac{\nu}{m} F}\right)^{s+r} \left(\dfrac{\bar{\beta}_1^{\,2}}{1 + \bar{\beta}_1^{\,2}}\right)^{s+t}}{s! \quad r! \quad t!}$$

$$\times \frac{\left(\frac{m+\nu}{2}\right)_{s+t} \left(\frac{\nu}{2}\right)_{s+r} \left(1 - \frac{m}{2} - t\right)_r}{\left(\frac{\nu+2}{2}\right)_{s+r}} =$$

$$(1 + \bar{\beta}_1^{\,2})^{\frac{m+\nu}{2}} \, (1 + \frac{\nu}{m} F)^{-\frac{m}{2}}$$

$$\times \sum_{r=0}^{\infty} \frac{\left(\frac{m+\nu}{2}\right)_r}{\left(\frac{\nu+2}{2}\right)_r} \cdot \left(\frac{\frac{\nu}{m} F}{1 + \frac{\nu}{m} F}\right)^r$$

From (3.1.17), (3.1.26), and (A.4) we obtain

$$(3.1.27\ \text{a-b}) \quad G(F) \rightarrow \frac{\left(\frac{\nu}{m} F\right)^{\frac{\nu}{2}}}{(\nu/2) B(\nu/2, m/2) (1 + \frac{\nu}{m} F)^{\frac{m+\nu}{2}}}$$

$$\times \; {}_2F_1 \left[\begin{matrix} 1, \; \frac{m+\nu}{2}; & \\ \frac{\nu+2}{2}; & \end{matrix} \; \frac{\frac{\nu}{m} F}{1 + \frac{\nu}{m} F} \right]$$

$$0 \leqq F < \infty$$

$$= 0 \quad \text{otherwise}$$

as $\bar{\mu}^2 \to \infty$.

From (3.1.7) and (3.1.27) we deduce that the distribution function of the identifiability test statistic converges to the (central) F distribution function with ν and m degrees of freedom as $\bar{\mu}^2 \to \infty$.

3.2 Some Computational Results

Let the moments (if they exist) of the identifiability test be defined by

$$(3.2.1) \quad E[F^h] = \int_{-\infty}^{\infty} F^h \, g(F) dF$$

where g(F) is specified in (3.1.4). Richardson (1968a) has obtained an expression for $E[F^h]$ and has shown that $h < \frac{m}{2}$ is a necessary and sufficient condition for $E[F^h]$ to exist.

$$(3.2.2) \quad E[F^h] = \left(\frac{m}{\nu}\right)^h e^{-\frac{\bar{\mu}^2}{2}(1 + \bar{\beta}_1^2)}$$

$$\times \left[\frac{\Gamma\left(\frac{m}{2} - h\right)\Gamma\left(\frac{\nu}{2} + h\right)\Gamma\left(\frac{\nu+1}{2} + h\right)\Gamma\left(\frac{\nu+2}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{\nu+1}{2}\right)\Gamma\left(\frac{\nu+2}{2} + h\right)} \right.$$

$$\times \sum_{j=0}^{\infty} \sum_{s=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{\left(\frac{\bar{\mu}^2}{2}\right)^j \left(\frac{\bar{\beta}_1 \bar{\mu}^2}{2}\right)^{s+\ell}}{j! \quad s! \quad \ell!}$$

$$\times \frac{\left(\frac{\nu+2}{2}\right)_{j+\ell}\left(\frac{1}{2}\right)_{j+\ell}\left(\frac{\nu}{2}+h\right)_s\left(\frac{\nu+1}{2}+h\right)_j}{\left(\frac{\nu+1}{2}\right)_{j+s+\ell}\left(\frac{1}{2}\right)_j\left(\frac{\nu+2}{2}+h\right)_{j+\ell}}$$

(Richardson, 1968a, p. 210).

The moments of a random variable distributed as (central) F with ν and m degrees of freedom are given by

$$(3.2.3) \quad E[F^h] = \left(\frac{m}{\nu}\right)^h \frac{\Gamma\left(\frac{m}{2} - h\right)\Gamma\left(\frac{\nu}{2} + h\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{\nu}{2}\right)}$$

with $h < \frac{m}{2}$.

When $m > 4$ we can use the method of moments (Kendall and Stuart, 1963, pp. 148-152) in order to determine a (central) F distribution which approximates G(F). Let the parameters of the approximating (central) F distribution be designated by $(\bar{\nu}, \bar{m})$. Let the first two moments of the identifiability test statistic be designated by μ_1 and μ_2. Then, making use of (3.2.3), we use the method of moments to obtain

$$(3.2.4\ \text{a-b}) \quad \bar{m} = \frac{2\mu_1}{\mu_1 - 1}$$

$$\bar{\nu} = \frac{2\mu_1^2}{2\mu_2 - \mu_1\mu_2 - \mu_1^2}$$

Computer programs in the Fortran IV language with double precision real variables have been written suitable for computing G(F) according to (3.1.9), for computing $\bar{G}(F)$ according to (3.1.7), and for approximating G(F) making use of (3.1.7), (3.2.2), and (3.2.4). In the following tables we present approximations of G(F) obtained by the method of moments. Results are presented for ν = 1, 2, 3, 4, 5, m = 10, $\bar{\beta}_1^2$ = 0, .25, 1, and $\bar{\mu}^2$ = 10. No results are presented where the computation is performed according to (3.1.9) since the computation time is prohibitively large. The inaccuracy in computing G(F) for large F in Table 3.2.6 with $\bar{\beta}_1^2$ = 1 is due to computational roundoff error resulting from the large number of terms required to satisfy the convergence criterion.

TABLE 3.2.1

Moments of the Identifiability Test Statistic:
$m = 10$, $\bar{\mu}^2 = 10$

ν	$\bar{\beta}_1^2 = 0$		$\bar{\beta}_1^2 = .25$		$\bar{\beta}_1^2 = 1$	
	E(F)	$E(F^2)$	E(F)	$E(F^2)$	E(F)	$E(F^2)$
1	1.139	5.281	1.222	6.316	1.387	9.063
2	1.149	3.572	1.235	4.253	1.417	6.098
3	1.157	3.013	1.245	3.572	1.438	5.115
4	1.164	2.739	1.252	3.234	1.454	4.621
5	1.170	2.579	1.258	3.032	1.465	4.320

TABLE 3.2.2

G(F): $m = 10,\ \bar{\mu}^2 = 10,\ \nu = 1$

F	$\bar{G}(F)$	$\bar{\beta}_1^2 = 0$		$\bar{\beta}_1^2 = .25$		$\bar{\beta}_1^2 = 1$	
		G(F)	G(F)-$\bar{G}$(F)	G(F)	G(F)-$\bar{G}$(F)	G(F)	G(F)-$\bar{G}$(F)
.2	.3357	.3958	.0601	.3705	.0348	.3180	-.0177
.4	.4587	.5098	.0511	.4875	.0288	.4415	-.0172
.6	.5433	.5864	.0431	.5671	.0238	.5269	-.0164
.8	.6073	.6435	.0362	.6266	.0193	.5923	-.0150
1.0	.6589	.6886	.0297	.6742	.0153	.6440	-.0149
1.2	.7005	.7261	.0256	.7128	.0123	.6867	-.0138
1.4	.7357	.7570	.0212	.7449	.0092	.7218	-.0139
1.6	.7650	.7830	.0180	.7726	.0076	.7513	-.0137
1.8	.7899	.8052	.0153	.7960	.0061	.7773	-.0126
2.0	.8120	.8254	.0133	.8160	.0040	.7990	-.0130
2.2	.8306	.8422	.0116	.8334	.0027	.8177	-.0129
2.4	.8468	.8569	.0102	.8495	.0027	.8350	-.0118
2.6	.8617	.8698	.0081	.8629	.0013	.8493	-.0123
2.8	.8742	.8822	.0080	.8748	.0006	.8627	-.0114
3.0	.8851	.8924	.0073	.8860	.0009	.8740	-.0112
3.5	.9083	.9132	.0049	.9077	-.0006	.8976	-.0107
4.0	.9259	.9302	.0042	.9249	-.0010	.9157	-.0102
4.5	.9387	.9433	.0046	.9375	-.0012	.9289	-.0098
5.0	.9495	.9529	.0034	.9483	-.0012	.9401	-.0093
5.5	.9580	.9613	.0033	.9569	-.0011	.9491	-.0089
6.0	.9639	.9673	.0034	.9630	-.0009	.9564	-.0075
6.5	.9696	.9728	.0032	.9689	-.0007	.9624	-.0072
7.0	.9742	.9766	.0024	.9726	-.0016	.9664	-.0078
7.5	.9770	.9805	.0035	.9768	-.0003	.9706	-.0064
8.0	.9803	.9829	.0026	.9801	-.0002	.9742	-.0062
8.5	.9831	.9857	.0026	.9821	-.0009	.9772	-.0059
9.0	.9844	.9872	.0028	.9847	.0003	.9787	-.0057
9.5	.9865	.9883	.0018	.9859	-.0006	.9810	-.0055

TABLE 3.2.3

G(F): m = 10, $\bar{\mu}^2$ = 10, ν = 2

F	$\bar{G}(F)$	$\bar{\beta}_1^2 = 0$ G(F)	$\bar{\beta}_1^2 = 0$ G(F)-$\bar{G}$(F)	$\bar{\beta}_1^2 = .25$ G(F)	$\bar{\beta}_1^2 = .25$ G(F)-$\bar{G}$(F)	$\bar{\beta}_1^2 = 1$ G(F)	$\bar{\beta}_1^2 = 1$ G(F)-$\bar{G}$(F)
.2	.1779	.2365	.0586	.2040	.0260	.1300	-.0479
.4	.3192	.3761	.0569	.3443	.0251	.2676	-.0516
.6	.4324	.4806	.0482	.4530	.0206	.3854	-.0469
.8	.5237	.5634	.0397	.5391	.0154	.4827	-.0410
1.0	.5975	.6296	.0321	.6094	.0119	.5624	-.0350
1.2	.6584	.6844	.0260	.6668	.0084	.6271	-.0313
1.4	.7086	.7290	.0204	.7138	.0052	.6808	-.0278
1.6	.7497	.7670	.0173	.7534	.0038	.7251	-.0245
1.8	.7845	.7982	.0137	.7860	.0015	.7620	-.0225
2.0	.8130	.8252	.0122	.8141	.0011	.7928	-.0202
2.2	.8377	.8475	.0098	.8372	-.0005	.8179	-.0198
2.4	.8578	.8663	.0085	.8576	-.0002	.8399	-.0179
2.6	.8757	.8833	.0076	.8743	-.0015	.8586	-.0172
2.8	.8910	.8971	.0061	.8893	-.0017	.8745	-.0164
3.0	.9034	.9098	.0064	.9015	-.0018	.8883	-.0151
3.5	.9278	.9335	.0058	.9259	-.0018	.9138	-.0140
4.0	.9458	.9503	.0045	.9432	-.0026	.9326	-.0132
4.5	.9579	.9614	.0035	.9566	-.0013	.9462	-.0116
5.0	.9666	.9703	.0037	.9657	-.0009	.9564	-.0102
5.5	.9739	.9768	.0029	.9725	-.0014	.9640	-.0098
6.0	.9786	.9816	.0030	.9775	-.0011	.9690	-.0096
6.5	.9821	.9853	.0032	.9813	-.0007	.9737	-.0084
7.0	.9847	.9881	.0034	.9842	-.0004	.9773	-.0073
7.5	.9876	.9902	.0027	.9865	-.0011	.9803	-.0073
8.0	.9890	.9919	.0029	.9882	-.0008	.9817	-.0073
8.5	.9902	.9932	.0031	.9895	-.0006	.9837	-.0064
9.0	.9910	.9933	.0024	.9906	-.0004	.9854	-.0056
9.5	.9916	.9942	.0026	.9914	-.0001	.9859	-.0057

TABLE 3.2.4

$G(F)$: $m = 10$, $\bar{\mu}^2 = 10$, $\nu = 3$

F	$\bar{G}(F)$	$\bar{\beta}_1^2 = 0$		$\bar{\beta}_1^2 = .25$		$\bar{\beta}_1^2 = 1$	
		G(F)	G(F)-$\bar{G}$(F)	G(F)	G(F)-$\bar{G}$(F)	G(F)	G(F)-$\bar{G}$(F)
.2	.1060	.1570	.0511	.1226	.0166	.0459	-.0600
.4	.2438	.3007	.0569	.2626	.0188	.1621	-.0817
.6	.3703	.4207	.0504	.3862	.0159	.2929	-.0774
.8	.4781	.5193	.0412	.4895	.0115	.4123	-.0658
1.0	.5674	.5996	.0323	.5751	.0077	.5130	-.0544
1.2	.6399	.6659	.0260	.6452	.0052	.5959	-.0440
1.4	.6997	.7203	.0206	.7025	.0029	.6626	-.0370
1.6	.7485	.7644	.0160	.7488	.0003	.7171	-.0314
1.8	.7884	.8014	.0131	.7876	-.0008	.7606	-.0278
2.0	.8211	.8314	.0103	.8195	-.0016	.7958	-.0253
2.2	.8473	.8570	.0097	.8460	-.0013	.8254	-.0219
2.4	.8697	.8775	.0078	.8672	-.0025	.8490	-.0207
2.6	.8883	.8955	.0071	.8858	-.0025	.8693	-.0190
2.8	.9039	.9096	.0057	.9013	-.0026	.8854	-.0185
3.0	.9169	.9225	.0055	.9144	-.0025	.8989	-.0181
3.5	.9401	.9453	.0053	.9380	-.0020	.9252	-.0149
4.0	.9561	.9614	.0053	.9546	-.0015	.9422	-.0139
4.5	.9671	.9717	.0046	.9660	-.0011	.9548	-.0123
5.0	.9747	.9786	.0038	.9732	-.0015	.9629	-.0118
5.5	.9802	.9842	.0041	.9791	-.0011	.9687	-.0114
6.0	.9841	.9875	.0035	.9834	-.0007	.9729	-.0111
6.5	.9860	.9898	.0038	.9857	-.0003	.9760	-.0100
7.0	.9882	.9914	.0032	.9882	.0000	.9783	-.0099
7.5	.9899	.9924	.0025	.9892	-.0007	.9809	-.0089
8.0	.9912	.9941	.0030	.9908	-.0004	.9822	-.0090
8.5	.9912	.9946	.0034	.9921	.0009	.9831	-.0081
9.0	.9920	.9949	.0029	.9922	.0001	.9839	-.0083
9.5	.9927	.9951	.0023	.9931	.0003	.9843	-.0085

TABLE 3.2.5

$G(F)$: $m = 10$, $\bar{\mu}^2 = 10$, $\nu = 4$

F	$\bar{G}(F)$	$\bar{\beta}_1^2 = 0$		$\bar{\beta}_1^2 = .25$		$\bar{\beta}_1^2 = 1$	
		G(F)	$G(F)-\bar{G}(F)$	G(F)	$G(F)-\bar{G}(F)$	G(F)	$G(F)-\bar{G}(F)$
.2	.0673	.1106	.0432	.0768	.0095	.0115	-.0558
.4	.1953	.2509	.0555	.2081	.0127	.0919	-.1034
.6	.3287	.3792	.0505	.3395	.0108	.2226	-.1061
.8	.4477	.4891	.0414	.4553	.0076	.3579	-.0899
1.0	.5480	.5804	.0324	.5524	.0044	.4763	-.0717
1.2	.6302	.6552	.0250	.6320	.0018	.5738	-.0564
1.4	.6967	.7161	.0194	.6965	-.0002	.6518	-.0449
1.6	.7503	.7650	.0148	.7488	-.0015	.7129	-.0374
1.8	.7934	.8054	.0120	.7910	-.0024	.7618	-.0316
2.0	.8282	.8383	.0102	.8253	-.0029	.8007	-.0275
2.2	.8563	.8644	.0081	.8531	-.0032	.8310	-.0253
2.4	.8792	.8865	.0074	.8759	-.0032	.8562	-.0230
2.6	.8978	.9047	.0069	.8946	-.0032	.8757	-.0221
2.8	.9131	.9197	.0066	.9100	-.0031	.8925	-.0206
3.0	.9257	.9313	.0056	.9228	-.0029	.9063	-.0193
3.5	.9480	.9537	.0057	.9457	-.0023	.9305	-.0176
4.0	.9632	.9681	.0049	.9614	-.0017	.9459	-.0172
4.5	.9723	.9767	.0044	.9711	-.0012	.9562	-.0161
5.0	.9792	.9831	.0039	.9776	-.0016	.9631	-.0161
5.5	.9832	.9865	.0033	.9821	-.0012	.9670	-.0162
6.0	.9859	.9897	.0038	.9851	-.0008	.9704	-.0155
6.5	.9886	.9920	.0034	.9872	-.0015	.9729	-.0157
7.0	.9898	.9927	.0029	.9896	-.0002	.9737	-.0161
7.5	.9905	.9940	.0035	.9906	.0001	.9750	-.0155
8.0	.9909	.9941	.0031	.9913	.0004	.9750	-.0159
8.5	.9911	.9949	.0038	.9918	.0006	.9758	-.0154
9.0	.9922	.9947	.0024	.9921	-.0002	.9754	-.0168
9.5	.9922	.9953	.0031	.9922	.0000	.9749	-.0173

TABLE 3.2.6

G(F): m = 10, $\bar{\mu}^2$ = 10, ν = 5

F	$\bar{G}(F)$	$\bar{\beta}_1^2 = 0$		$\bar{\beta}_1^2 = .25$		$\bar{\beta}_1^2 = 1$	
		G(F)	G(F)-$\bar{G}$(F)	G(F)	G(F)-$\bar{G}$(F)	G(F)	G(F)-$\bar{G}$(F)
.2	.0448	.0809	.0361	.0494	.0047	.0014	-.0434
.4	.1618	.2147	.0529	.1692	.0074	.0459	-.1159
.6	.2979	.3486	.0506	.3045	.0066	.1661	-.1318
.8	.4251	.4670	.0419	.4295	.0044	.3123	-.1128
1.0	.5344	.5665	.0321	.5363	.0020	.4462	-.0882
1.2	.6235	.6480	.0245	.6236	.0001	.5562	-.0673
1.4	.6951	.7141	.0190	.6938	-.0013	.6427	-.0524
1.6	.7528	.7665	.0137	.7499	-.0030	.7090	-.0438
1.8	.7981	.8092	.0111	.7945	-.0036	.7609	-.0373
2.0	.8341	.8436	.0095	.8311	-.0030	.8011	-.0330
2.2	.8628	.8712	.0085	.8597	-.0031	.8326	-.0301
2.4	.8865	.8936	.0071	.8827	-.0039	.8567	-.0298
2.6	.9049	.9116	.0067	.9012	-.0038	.8765	-.0285
2.8	.9198	.9254	.0057	.9162	-.0036	.8915	-.0283
3.0	.9318	.9375	.0057	.9293	-.0024	.9044	-.0274
3.5	.9535	.9585	.0051	.9507	-.0027	.9256	-.0279
4.0	.9669	.9715	.0046	.9648	-.0021	.9384	-.0286
4.5	.9755	.9797	.0042	.9739	-.0016	.9453	-.0302
5.0	.9812	.9850	.0038	.9800	-.0012	.9503	-.0309
5.5	.9850	.9885	.0035	.9842	-.0008	.9525	-.0325
6.0	.9876	.9908	.0032	.9861	-.0015	.9535	-.0340
6.5	.9884	.9923	.0040	.9882	-.0002	.9529	-.0355
7.0	.9897	.9934	.0037	.9897	.0000	.9528	-.0369
7.5	.9906	.9941	.0035	.9899	-.0007	.9514	-.0392
8.0	.9913	.9947	.0034	.9908	-.0005	.9500	-.0414
8.5	.9908	.9950	.0042	.9905	-.0004	.9483	-.0425
9.0	.9912	.9943	.0031	.9910	-.0002	.9457	-.0455
9.5	.9915	.9945	.0030	.9915	-.0000	.9441	-.0474

CHAPTER 4

THE DISTRIBUTION FUNCTIONS FOR THE GCL STRUCTURAL VARIANCE ESTIMATORS

4.1 The Asymptotic Distribution Function Associated with V_2

We denote the distribution function of U corresponding to (1.2.6 c) by $F_2(U)$ with associated parameter space $(\nu, \bar{\beta}_1^{\,2}, \bar{\mu}^2)$. The chi-square distribution function with ν degrees of freedom is denoted by $\bar{F}_2(U)$.

Basmann and Richardson (1969b) have derived the exact finite sample distribution function $F_2(U)$.

(4.1.1 a-b)
$$F_2(U) = \frac{\left[\left(1 + \bar{\beta}_1^{\,2}\right)\frac{U}{2}\right]^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu+2}{2}\right)}$$

$$\times \sum_{s=0}^{\infty} \frac{\left(\frac{\nu}{2}\right)_s \left[-\left(1 + \bar{\beta}_1^{\,2}\right)\frac{U}{2}\right]^s}{\left(\frac{\nu+2}{2}\right)_s \, s!}$$

$$\times \left\{e^{-\frac{\bar{\mu}^2}{2}} \, \Phi_2\left(\frac{\nu+1}{2}, \frac{\nu}{2} + s; \frac{\nu+1}{2}; \frac{\bar{\mu}^2}{2}, -\frac{\bar{\beta}_1^{\,2}\bar{\mu}^2}{2}\right)\right\}$$

$$0 \leqq U < \infty$$

$$= 0 \quad \text{otherwise}$$

(Basmann and Richardson, 1969b, p. 24). $\bar{F}_2(U)$ is given by

$$(4.1.2\text{ a-b})\quad \bar{F}_2(U) = \frac{\left(\frac{U}{2}\right)^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu+2}{2}\right)} \sum_{s=0}^{\infty} \frac{\left(\frac{\nu}{2}\right)_s \left(-\frac{U}{2}\right)^s}{\left(\frac{\nu+2}{2}\right)_s s!}$$

$$0 \leqq U < \infty$$

$$= 0 \quad \text{otherwise}.$$

Designating the expression in curled braces in (4.1.1 a) by P(s) we use (A.22) to deduce

$$(4.1.3)\quad P(s) \rightarrow (1 + \bar{\beta}_1^{\,2})^{-\frac{\nu}{2} - s}$$

as $\bar{\mu}^2 \rightarrow \infty$.

Then by (4.1.1), (4.1.2), and (4.1.3) we obtain

$$(4.1.4)\quad F_2(U) \rightarrow \bar{F}_2(U)$$

as $\bar{\mu}^2 \rightarrow \infty$.

4.2 The Asymptotic Distribution Function Associated with V_3

We denote the distribution function of U corresponding to (1.2.6 b) by $F_3(U)$ with associated parameter space $(\nu, m, \bar{\beta}_1^{\,2}, \bar{\mu}^2)$. The chi-square distribution function with m degrees of freedom is denoted by $\bar{F}_3(U)$.

Basmann and Richardson (1969 a) have derived the exact finite sample distribution function $F_3(U)$.

$$(4.2.1\ \text{a-b})\quad F_3(U) = \frac{\left[\left(1 + \bar{\beta}_1^{\,2}\right)\frac{U}{2}\right]^{\frac{m}{2}}}{\Gamma\left(\frac{m+2}{2}\right)}$$

$$\times \left[\frac{\Gamma\left(\frac{\nu+2}{2}\right)\Gamma\left(\frac{m+\nu+1}{2}\right)}{\Gamma\left(\frac{\nu+1}{2}\right)\Gamma\left(\frac{m+\nu+2}{2}\right)} \right]$$

$$\times \sum_{k=0}^{\infty} \frac{\left(\frac{m}{2}\right)_k \left(\frac{m+\nu+1}{2}\right)_k \left[-\left(1 + \bar{\beta}_1^{\,2}\right)\frac{U}{2}\right]^k}{\left(\frac{m+2}{2}\right)_k \left(\frac{m+\nu+2}{2}\right)_k \quad k!}$$

$$\times \left\{ e^{-\frac{\bar{\mu}^2}{2}} \sum_{j=0}^{\infty} \frac{\left(\frac{1}{2}\right)_j \left(\frac{m}{2} + k\right)_j}{\left(\frac{\nu+1}{2}\right)_j \left(\frac{m+\nu+2}{2} + k\right)_j} \cdot \frac{\left(-\frac{\bar{\beta}_1^{\,2}\bar{\mu}^2}{2}\right)^j}{j!}\right.$$

$$\left. \times\ {}_3F_3 \left[\begin{matrix} \frac{\nu+2}{2}, & \frac{1}{2} + j, & \frac{m+\nu+1}{2} + k; \\ \frac{1}{2}, & \frac{\nu+1}{2} + j, & \frac{m+\nu+2}{2} + k + j; \end{matrix} \;\; \frac{\bar{\mu}^2}{2} \right] \right\}$$

$$0 \leqq U < \infty$$

$$= 0 \quad \text{otherwise}$$

(Basmann and Richardson, 1969a, p. 20). $\bar{F}_3(U)$ is given by

$$(4.2.2\text{ a-b})\quad \bar{F}_3(U) = \frac{\left(\frac{U}{2}\right)^{\frac{m}{2}}}{\Gamma\left(\frac{m+2}{2}\right)} \sum_{k=0}^{\infty} \frac{\left(\frac{m}{2}\right)_k \left(-\frac{U}{2}\right)^k}{\left(\frac{m+2}{2}\right)_k k!}$$

$$0 \leqq U < \infty$$

$$= 0 \quad \text{otherwise.}$$

Designate the expression in curled braces in (4.2.1 a) by S(k). By use of (A.2) and (A.4) we have

$$(4.2.3)\quad S(k) = e^{-\frac{\bar{\mu}^2}{2}} \sum_{p=0}^{\infty} \frac{\left(\frac{\bar{\mu}^2}{2}\right)^p}{p!}$$

$$\times \frac{\left(\frac{\nu+2}{2}\right)_p \left(\frac{m+\nu+1}{2} + k\right)_p}{\left(\frac{\nu+1}{2}\right)_p \left(\frac{m+\nu+2}{2} + k\right)_p}$$

$$\times\ {}_2F_2 \left[\begin{matrix} \frac{m}{2} + k,\ \frac{1}{2} + p; \\ \frac{\nu+1}{2} + p,\ \frac{m+\nu+2}{2} + k + p; \end{matrix} \quad - \frac{\bar{\beta}_1^2 \bar{\mu}^2}{2} \right]$$

By (A.2), (A.4), and (A.13) we can write (4.2.3) as

$$(4.2.4)\quad S(k) = e^{-\frac{\bar{\mu}^2}{2}} \sum_{n=0}^{\infty} \frac{\left(\frac{m}{2} + k\right)_n}{\left(\frac{\nu+1}{2}\right)_n} \cdot \frac{\left(-\frac{\bar{\beta}_1^2 \bar{\mu}^2}{2}\right)^n}{n!}$$

$$\times \sum_{q=0}^{\infty} \frac{\left(\frac{m}{2}+k+n\right)_q \left(\frac{m+\nu+1}{2}+k\right)_q}{\left(\frac{\nu+1}{2}+n\right)_q \left(\frac{m+\nu+2}{2}+k\right)_q} \cdot \frac{\left(\frac{\bar{\beta}_1^2 \bar{\mu}^2}{2}\right)^q}{q!}$$

$$\times \; {}_2F_2 \left[\begin{matrix} \frac{\nu+2}{2}, \frac{m+\nu+1}{2}+k; \\ \frac{\nu+1}{2}+n+q, \frac{m+\nu+2}{2}+k+q; \end{matrix} \; \frac{\bar{\mu}^2}{2} \right]$$

Now, by (A.18) we have

$$(4.2.5) \quad \lim_{\bar{\mu}^2 \to \infty} S(k) = \lim_{\bar{\mu}^2 \to \infty} e^{-\frac{\bar{\mu}^2}{2}} \sum_{n=0}^{\infty} \frac{\left(-\frac{\bar{\beta}_1^2 \bar{\mu}^2}{2}\right)^n}{n!}$$

$$\times \frac{\left(\frac{m}{2}+k\right)_n}{\left(\frac{\nu+1}{2}\right)_n} \; {}_2F_2 \left[\begin{matrix} \frac{\nu+2}{2}, \frac{m+\nu+1}{2}+k; \\ \frac{\nu+1}{2}+n, \frac{m+\nu+2}{2}+k; \end{matrix} \; \frac{\bar{\mu}^2}{2} \right]$$

Then by use of (A.2), (A.4), and (A.23) we obtain

$$(4.2.6) \quad \lim_{\bar{\mu}^2 \to \infty} S(k) = \lim_{\bar{\mu}^2 \to \infty} e^{-\frac{\bar{\mu}^2}{2}}$$

$$\times \sum_{p=0}^{\infty} \frac{\left(\frac{m}{2}+k\right)_p \left(\frac{1}{2}\right)_p}{\left(\frac{\nu+1}{2}\right)_p \left(\frac{m+\nu+2}{2}+k\right)_p} \cdot \frac{\left(\frac{\bar{\mu}^2}{2}\right)^p}{p!}$$

$$\times \quad \Phi_2 \left(\frac{\nu+1}{2}, \frac{m}{2} + k; \frac{\nu+1}{2} + p; \frac{\bar{\mu}^2}{2}, \quad - \frac{\bar{\beta}_1^{\,2} \bar{\mu}^2}{2} \right)$$

By (A.2), (A.4), and (A.22) we then obtain

$$(4.2.7) \quad S(k) \to (1 + \bar{\beta}_1^{\,2})^{-\frac{m}{2} - k} \; {}_2F_1 \left[\begin{matrix} \frac{m}{2} + k, \frac{1}{2} ; \\ \\ \frac{m+\nu+2}{2} + k ; \end{matrix} \; 1 \right]$$

as $\bar{\mu}^2 \to \infty$.

By (A.5) we can write (4.2.7) as

$$(4.2.8) \quad S(k) \to (1 + \bar{\beta}_1^{\,2})^{-\frac{m}{2} - k} \left[\frac{\Gamma\left(\frac{m+\nu+2}{2} + k\right)\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu+2}{2}\right)\Gamma\left(\frac{m+\nu+1}{2} + k\right)} \right]$$

as $\bar{\mu}^2 \to \infty$.

Then from (4.2.1), (4.2.2), (4.2.8), and (A.2) we can deduce

$$(4.2.9) \quad F_3(U) \to \bar{F}_3(U)$$

as $\bar{\mu}^2 \to \infty$.

4.3 The Asymptotic Distribution Function Associated with V_4

We denote the distribution function of U corresponding to (1.2.6 a) by $F_4(U)$ with associated parameter space $(\nu, m, \bar{\beta}_1^{\,2}, \bar{\mu}^2)$.

The chi-square distribution function with m+ν degrees of freedom is denoted by $\overline{F}_4(U)$.

Basmann and Richardson have derived the exact finite sample distribution function $F_4(U)$.

$$\text{(4.3.1 a-b)} \quad F_4(U) = \frac{\left[\left(1 + \overline{\beta}_1^{\,2}\right)\frac{U}{2}\right]^{\frac{m+\nu}{2}}}{\Gamma\left(\frac{m+\nu+2}{2}\right)}$$

$$\times \left[\frac{\Gamma\left(\frac{\nu+2}{2}\right)\Gamma\left(\frac{m+\nu+1}{2}\right)}{\Gamma\left(\frac{\nu+1}{2}\right)\Gamma\left(\frac{m+\nu+2}{2}\right)} \right]$$

$$\times \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{m+\nu}{2}\right)_{n+r}}{\left(\frac{m+\nu}{2}\right)_n \left(\frac{m+\nu+2}{2}\right)_{n+r}} \cdot \frac{(-)^r\left[\left(1 + \beta_1^{\,2}\right)\frac{U}{2}\right]^{n+r}}{n! \quad r!}$$

$$\times \sum_{k=0}^{\infty} \frac{(-n)_k \left(\frac{\nu+2}{2}\right)_k}{\left(\frac{m+\nu+2}{2}\right)_k \quad k!}$$

$$\times \left\{ e^{-\frac{\overline{\mu}^2}{2}} \sum_{s=0}^{\infty} \sum_{q=0}^{\infty} \frac{\left(\frac{m}{2}\right)_q \left(\frac{1}{2} + k\right)_q \left(\frac{\nu}{2}\right)_s \left(\frac{m+\nu}{2} + n + r\right)_s}{\left(\frac{m+\nu+2}{2} + k\right)_q \left(\frac{m+\nu}{2} + n\right)_s \left(\frac{\nu+1}{2}\right)_{s+q}} \right.$$

$$\times \frac{\left(-\frac{\overline{\beta}_1^{\,2}\overline{\mu}^2}{2}\right)^{s+q}}{s! \quad q!} \; {}_3F_3 \left[\begin{matrix} \frac{\nu+2}{2} + k, \ \frac{1}{2} + q, \ \frac{m+\nu+1}{2} \ ; \\ \frac{1}{2}, \ \frac{\nu+1}{2} + s + q, \ \frac{m+\nu+2}{2} + k + q ; \end{matrix} \ \frac{\overline{\mu}^2}{2} \right]$$

$$0 \leqq U < \infty$$

$$= 0 \quad \text{otherwise}$$

(Basmann, Ebbeler, and Richardson, 1970, pp. 19-20).

$\overline{F}_4(U)$ is given by

$$\text{(4.3.2 a-b)} \quad \overline{F}_4(U) = \frac{\left(\frac{U}{2}\right)^{\frac{m+\nu}{2}}}{\Gamma\left(\frac{m+\nu+2}{2}\right)} \sum_{r=0}^{\infty} \frac{\left(\frac{m+\nu}{2}\right)_r \left(-\frac{U}{2}\right)^r}{\left(\frac{m+\nu+2}{2}\right)_r \, r!}$$

$$0 \leqq U < \infty$$

$$= 0 \quad \text{otherwise}.$$

Designate the expression in curled braces in (4.3.1a) by T(n,r,k). By use of (A.2) and (A.4) we have

$$\text{(4.3.3)} \quad T(n,r,k) = e^{-\frac{\bar{\mu}^2}{2}} \sum_{t=0}^{\infty} \sum_{s=0}^{\infty} \frac{\left(\frac{\bar{\mu}^2}{2}\right)^t}{t!} \frac{\left(-\frac{\bar{\beta}_1^2 \bar{\mu}^2}{2}\right)^s}{s!}$$

$$\times \frac{\left(\frac{\nu+2}{2}+k\right)_t \left(\frac{m+\nu+1}{2}\right)_t \left(\frac{\nu}{2}\right)_s \left(\frac{m+\nu}{2}+n+r\right)_s}{\left(\frac{m+\nu+2}{2}+k\right)_t \left(\frac{\nu+1}{2}\right)_t \left(\frac{m+\nu}{2}+n\right)_s \left(\frac{\nu+1}{2}+t\right)_s}$$

$$\times \ {}_3F_3 \left[\begin{matrix} \frac{m}{2}, \ \frac{1}{2}+k, \ \frac{1}{2}+t; \\ \frac{1}{2}, \ \frac{m+\nu+2}{2}+k+t, \ \frac{\nu+1}{2}+t+s; \end{matrix} \ -\frac{\bar{\beta}_1^2 \bar{\mu}^2}{2} \right]$$

By (A.2), (A.4), and (A.13) we can write (4.3.3) as

$$(4.3.4)\quad T(n,r,k) = e^{-\frac{\bar{\mu}^2}{2}} \sum_{s=0}^{\infty} \sum_{q=0}^{\infty} \frac{\left(-\frac{\bar{\beta}_1^2\bar{\mu}^2}{2}\right)^{s+q}}{s!\quad q!}$$

$$\times \frac{\left(\frac{\nu}{2}\right)_s \left(\frac{m+\nu}{2} + n + r\right)_s \left(\frac{m}{2}\right)_q \left(\frac{1}{2} + k\right)_q}{\left(\frac{m+\nu}{2} + n\right)_s \left(\frac{1}{2}\right)_q \left(\frac{\nu+1}{2}\right)_{s+q}}$$

$$\times \sum_{p=0}^{\infty} \frac{\left(\frac{m+\nu+1}{2} + k\right)_p \left(\frac{m}{2} + q\right)_p \left(\frac{1}{2} + k + q\right)_p}{\left(\frac{m+\nu+2}{2} + k\right)_p \left(\frac{1}{2} + q\right)_p \left(\frac{\nu+1}{2} + s + q\right)_p}$$

$$\times \frac{\left(\frac{\bar{\beta}_1^2\bar{\mu}^2}{2}\right)^p}{p!} \quad {}_2F_2\left[\begin{matrix} \frac{\nu+2}{2} + k,\ \frac{m+\nu+1}{2}; \\ \frac{m+\nu+2}{2} + k + p,\ \frac{\nu+1}{2} + s + q + p; \end{matrix} \ \frac{\bar{\mu}^2}{2}\right]$$

Now, by (A.2), (A.4), and (A.18) we obtain from (4.3.4)

$$(4.3.5)\quad \lim_{\bar{\mu}^2 \to \infty} T(n,r,k) = \lim_{\bar{\mu}^2 \to \infty} e^{-\frac{\bar{\mu}^2}{2}}$$

$$\times \sum_{t=0}^{\infty} \sum_{s=0}^{\infty} \sum_{q=0}^{\infty} \frac{\left(\frac{\bar{\mu}^2}{2}\right)^t \left(-\frac{\bar{\beta}_1^2\bar{\mu}^2}{2}\right)^{s+q}}{t!\quad s!\quad q!}$$

$$\times \frac{\left(\frac{\nu+2}{2}+k\right)_t\left(\frac{m+\nu+1}{2}\right)_t\left(\frac{\nu}{2}\right)_s\left(\frac{m+\nu}{2}+n+r\right)_s\left(\frac{m}{2}\right)_q\left(\frac{1}{2}+k\right)_q}{\left(\frac{m+\nu+2}{2}+k\right)_t\left(\frac{m+\nu}{2}+n\right)_s\left(\frac{1}{2}\right)_q\left(\frac{\nu+1}{2}\right)_{t+s+q}}$$

By (A.29) we can write (4.3.5) as

$$(4.3.6)\quad \lim_{\bar{\mu}^2\to\infty} T(n,r,k) = \lim_{\bar{\mu}^2\to\infty} e^{-\frac{\bar{\mu}^2}{2}}$$

$$\times \sum_{t=0}^{\infty}\sum_{s=0}^{\infty}\sum_{q=0}^{\infty} \frac{\left(\frac{\mu^2}{2}\right)^t\left(-\frac{\bar{\beta}_1^2\bar{\mu}^2}{2}\right)^{s+q}}{t!\quad s!\quad q!}$$

$$\times \frac{\left(\frac{m}{2}\right)_t\left(\frac{1}{2}+k\right)_t\left(\frac{m}{2}+n\right)_s(-r)_s\left(\frac{1-m}{2}\right)_q(-k)_q}{\left(\frac{m+\nu+2}{2}+k\right)_t\left(\frac{m+\nu}{2}+n\right)_s\left(\frac{1}{2}\right)_q\left(\frac{\nu+1}{2}\right)_{t+s+q}}$$

$$\times \Phi_2^3\left(\frac{\nu+1}{2},\ \frac{\nu}{2}+r,\ \frac{m}{2}+k;\ \frac{\nu+1}{2}+t+s+q;\ \frac{\bar{\mu}^2}{2},\ -\frac{\bar{\beta}_1^2\bar{\mu}^2}{2},\ -\frac{\bar{\beta}_1^2\bar{\mu}^2}{2}\right)$$

By (A.2), (A.4), and (A.28) we have from (4.3.6)

$$(4.3.7)\quad T(n,r,k) \to (1+\bar{\beta}_1^2)^{-\frac{m+\nu}{2}-r-k}$$

$$\times \; {}_2F_1\left[\begin{matrix} \frac{m}{2}, \frac{1}{2}+k; \\ \frac{m+\nu+2}{2}+k; \end{matrix} \; 1\right] \; {}_2F_1\left[\begin{matrix} -r, \frac{m}{2}+n; \\ \frac{m+\nu}{2}+n; \end{matrix} \; -\bar{\beta}_1^{\,2}\right]$$

$$\times \; {}_2F_1\left[\begin{matrix} -k, \frac{1-m}{2}; \\ \frac{1}{2}; \end{matrix} \; -\bar{\beta}_1^{\,2}\right]$$

as $\bar{\mu}^2 \to \infty$.

By (A.5) we can write (4.3.7) as

$$(4.3.8) \quad T(n,r,k) \to (1+\bar{\beta}_1^{\,2})^{-\frac{m+\nu}{2}-r-k}$$

$$\times \left[\frac{\Gamma\left(\frac{m+\nu+2}{2}+k\right)\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu+2}{2}+k\right)\Gamma\left(\frac{m+\nu+1}{2}\right)}\right]$$

$$\times \; {}_2F_1\left[\begin{matrix} -r, \frac{m}{2}+n; \\ \frac{m+\nu}{2}+n; \end{matrix} \; -\bar{\beta}_1^{\,2}\right] \; {}_2F_1\left[\begin{matrix} -k, \frac{1-m}{2}; \\ \frac{1}{2}; \end{matrix} \; -\bar{\beta}_1^{\,2}\right]$$

as $\bar{\mu}^2 \to \infty$.

Then from (4.3.1), (4.3.8), and (A.2) we obtain

$$(4.3.9\text{ a-b}) \quad F_4(U) \to \frac{\left(\frac{U}{2}\right)^{\frac{m+\nu}{2}}}{\Gamma\left(\frac{m+\nu+2}{2}\right)}$$

$$\times \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{m+\nu}{2}\right)_{n+r}}{\left(\frac{m+\nu}{2}\right)_n \left(\frac{m+\nu+2}{2}\right)_{n+r}} \cdot \frac{(-)^r (1+\bar{\beta}_1^{\,2})^n \left(\frac{U}{2}\right)^{r+n}}{r! \quad n!}$$

$$\times \sum_{k=0}^{\infty} \frac{(-n)_k (1+\bar{\beta}_1^{\,2})^{-k}}{k!} \quad {}_2F_1\left[\begin{matrix} -r, \frac{m}{2}+n; \\ \frac{m+\nu}{2}+n; \end{matrix} \; -\bar{\beta}_1^{\,2}\right]$$

$$\times \quad {}_2F_1\left[\begin{matrix} -k, \frac{1-m}{2}; \\ \frac{1}{2}; \end{matrix} \; -\bar{\beta}_1^{\,2}\right]$$

$0 \leqq U < \infty$

$= 0$ otherwise

as $\bar{\mu}^2 \to \infty$.

By (A.1) and (A.4) we have

$$(4.3.10) \quad \sum_{k=0}^{\infty} \frac{(-n)_k (1+\bar{\beta}_1^{\,2})^{-k}}{k!} \quad {}_2F_1\left[\begin{matrix} -k, \frac{1-m}{2}; \\ \frac{1}{2}; \end{matrix} \; -\bar{\beta}_1^{\,2}\right]$$

$$= \sum_{k=0}^{\infty} \frac{(-n)_k (1+\bar{\beta}_1^{\,2})^{-k}}{k!} \sum_{p=0}^{k} \frac{(-k)_p \left(\frac{1-m}{2}\right)_p}{\left(\frac{1}{2}\right)_p} \cdot \frac{\left(-\bar{\beta}_1^{\,2}\right)^p}{p!}$$

Replacing k by k+p on the right side in (4.3.10) and making use of (A.7) we obtain

$$(4.3.11)\quad \sum_{k=0}^{\infty} \frac{(-n)_k (1+\bar{\beta}_1^2)^{-k}}{k!} \; {}_2F_1\left[\begin{matrix} -k, \frac{1-m}{2} ; \\ \frac{1}{2} ; \end{matrix} \; -\bar{\beta}_1^2 \right]$$

$$= \sum_{p=0}^{\infty} \frac{(-n)_p (\frac{1-m}{2})_p}{\left(\frac{1}{2}\right)_p} \cdot \frac{\left(\frac{\bar{\beta}_1^2}{1+\bar{\beta}_1^2}\right)^p}{p!}$$

$$\times \sum_{k=0}^{\infty} \frac{(-n+p)_k (1+\bar{\beta}_1^2)^{-k}}{k!}$$

From (A.6) and (4.3.11) we have

$$(4.3.12)\quad \sum_{k=0}^{\infty} \frac{(-n)_k (1+\bar{\beta}_1^2)^{-k}}{k!} \; {}_2F_1\left[\begin{matrix} -k, \frac{1-m}{2} ; \\ \frac{1}{2} ; \end{matrix} \; -\bar{\beta}_1^2 \right]$$

$$= \left(\frac{\bar{\beta}_1^2}{1+\bar{\beta}_1^2}\right)^n \sum_{p=0}^{\infty} \frac{(-n)_p (\frac{1-m}{2})_p}{\left(\frac{1}{2}\right)_p \, p!}$$

By (A.4), (A.8), and (4.3.12) we can write (4.3.9) as

(4.3.13 a-b) $$F_4(U) \to \frac{\left(\frac{U}{2}\right)^{\frac{m+\nu}{2}}}{\Gamma\left(\frac{m+\nu+2}{2}\right)}$$

$$\times \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{\left(\frac{m}{2}\right)_n \left(\frac{m+\nu}{2}\right)_{n+r}}{\left(\frac{m+\nu}{2}\right)_n \left(\frac{m+\nu+2}{2}\right)_{n+r}} \cdot \frac{(-)^r (\bar{\beta}_1^{\,2})^n \left(\frac{U}{2}\right)^{r+n}}{r! \quad n!}$$

$$\times \sum_{s=0}^{\infty} \frac{(-r)_s (\frac{m}{2} + n)_s}{\left(\frac{m+\nu}{2} + n\right)_s} \cdot \frac{\left(- \bar{\beta}_1^{\,2}\right)^s}{s!}$$

$$0 \leqq U < \infty$$

$$= 0 \quad \text{otherwise}$$

as $\bar{\mu}^{2} \to \infty$.

Replacing r by r-n in (4.3.13 a) and making use of (A.2) and (A.7) we write (4.3.13) as

(4.3.14 a-b) $$F_4(U) \to \frac{\left(\frac{U}{2}\right)^{\frac{m+\nu}{2}}}{\Gamma\left(\frac{m+\nu+2}{2}\right)}$$

$$\times \sum_{r=0}^{\infty} \frac{\left(\frac{m+\nu}{2}\right)_r \left(- \frac{U}{2}\right)^r}{\left(\frac{m+\nu+2}{2}\right)_r \quad r!}$$

$$\times \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-r)_{n+s}\left(\frac{m}{2}\right)_{n+s}}{\left(\frac{m+\nu}{2}\right)_{n+s}} \cdot \frac{(-)^s(\bar{\beta}_1^{\,2})^{n+s}}{n!\ \ s!}$$

$$0 \leqq U < \infty$$

$$= 0 \quad \text{otherwise}$$

as $\bar{\mu}^2 \to \infty$.

Replacing s by s-n in (4.3.14 a) and making use of (A.7) we write (4.3.14) as

$$(4.3.15\ \text{a-b}) \quad F_4(U) \to \frac{\left(\frac{U}{2}\right)^{\frac{m+\nu}{2}}}{\Gamma\left(\frac{m+\nu+2}{2}\right)}$$

$$\times \sum_{r=0}^{\infty} \frac{\left(\frac{m+\nu}{2}\right)_r \left(-\frac{U}{2}\right)^r}{\left(\frac{m+\nu+2}{2}\right)_r \ r!}$$

$$\times \sum_{s=0}^{\infty} \frac{(-r)_s\left(\frac{m}{2}\right)_s}{\left(\frac{m+\nu}{2}\right)_s} \cdot \frac{\left(-\bar{\beta}_1^{\,2}\right)^s}{s!} \sum_{n=0}^{\infty} \frac{(-s)_n}{n!}$$

$$0 \leqq U < \infty$$

$$= 0 \quad \text{otherwise}$$

as $\bar{\mu}^2 \to \infty$.

From (4.3.1), (4.3.2), (4.3.15), and (A.16) we can deduce

$$(4.3.16)\quad F_4(U) \to \overline{F}_4(U)$$

as $\overline{\mu}^2 \to \infty$.

CHAPTER 5

APPROXIMATIONS OF THE DISTRIBUTION FUNCTIONS FOR THE GCL STRUCTURAL VARIANCE ESTIMATORS

5.1 Approximations of the Exact Finite Sample Distribution Function Associated with $\underline{V}_2$

Ebbeler and McDonald (1969) have derived the following expression for $F_2(U) - \bar{F}_2(U)$ making use of an asymptotic expansion for confluent hypergeometric functions due to Slater (Slater, 1960, p. 60):

(5.1.1 a-b) $F_2(U) - \bar{F}_2(U) \simeq$

$$\left(\frac{\bar{\beta}_1^{\,2}}{1 + \bar{\beta}_1^{\,2}}\right) \frac{\left(\frac{U}{2}\right)^{\frac{\nu}{2}} e^{-\frac{U}{2}} (\nu-1)}{\bar{\mu}^2 \; \Gamma\left(\frac{\nu}{2}\right)}$$

$$\times \sum_{n=0}^{S-2} \left(\frac{3-\nu}{2}\right)_n \left(\frac{\bar{\mu}^2}{2}\right)^{-n}$$

$$\times \sum_{r=0}^{n} \frac{(-n)_r \left(\frac{\nu+2}{2}\right)_r}{(2)_r} \frac{\left(\frac{\bar{\beta}_1^{\,2}}{1 + \bar{\beta}_1^{\,2}}\right)^r}{r!}$$

$$\times \sum_{s=0}^{r} \frac{(-r)_s \left(\frac{U}{2}\right)^s}{\left(\frac{\nu+2}{2}\right)_s s!}$$

$$0 \leqq U < \infty$$

$$= 0 \quad \text{otherwise}$$

(Ebbeler and McDonald, 1969, p. 8).

From (5.1.1) we obtain

(5.1.2 a-b) $\frac{d}{dU} \left[F_2(U) - \bar{F}_2(U)\right] \simeq$

$$\left(\frac{\bar{\beta}_1^{\,2}}{1 + \bar{\beta}_1^{\,2}}\right) \frac{(\nu-1)\left(\frac{U}{2}\right)^{\frac{\nu}{2} - 1} e^{-\frac{U}{2}}}{4\, \bar{\mu}^2\, \Gamma\left(\frac{\nu}{2}\right)}$$

$$\times \sum_{n=0}^{S-2} \left(\frac{3-\nu}{2}\right)_n \left(\frac{\bar{\mu}^2}{2}\right)^{-n}$$

$$\times \left[\nu \sum_{r=0}^{n} \frac{(-n)_r \left(\frac{\nu+2}{2}\right)_r \left(\frac{\bar{\beta}_1^{\,2}}{1 + \bar{\beta}_1^{\,2}}\right)^r}{(2)_r \, r!} \right.$$

$$\times \sum_{s=0}^{r} \frac{(-r)_s \left(\frac{U}{2}\right)^s}{\left(\frac{\nu+2}{2}\right)_s s!}$$

$$- U \sum_{r=0}^{n} \frac{(-n)_r \left(\frac{\nu+4}{2}\right)_r}{(2)_r} \frac{\left(\frac{\bar{\beta}_1^2}{1+\bar{\beta}_1^2}\right)^r}{r!}$$

$$\times \sum_{s=0}^{r} \frac{(-r)_s \left(\frac{U}{2}\right)^s}{\left(\frac{\nu+4}{2}\right)_s s!} \Big]$$

$$0 \leqq U < \infty$$

$$= 0 \quad \text{otherwise}$$

(Ebbeler and McDonald, 1969, p. 9).

From (5.1.2), if we neglect powers of $\left(\frac{1}{\mu^2}\right)$ greater than one, the function $F_2(U) - \bar{F}_2(U)$ achieves an extreme value, which must be a maximum, for $U \simeq \nu$.

We therefore base a decision on approximating $F_2(U)$ by $\bar{F}_2(U)$ on the value of $F_2(\nu) - \bar{F}_2(\nu)$; i.e. when $F_2(\nu) - \bar{F}_2(\nu)$ is sufficiently small (an arbitrary specification) we approximate $F_2(U)$ by $\bar{F}_2(U)$. We note that $F_2(\nu) - \bar{F}_2(\nu)$ is never negative. Basmann and Richardson (1969b, pp. 29-31) have shown that $F_2(U) \geqq \bar{F}_2(U)$ for all $U \geqq 0$.

Ebbeler and McDonald (1969) have investigated alternative approximations to $F_2(U)$ for those cases when $F_2(\nu) - \bar{F}_2(\nu)$ has an unacceptable value. The two methods used were the method of moments (Kendall and Stuart, 1963, pp. 148-152) and what was termed the modified method of moments (Ebbeler and McDonald, 1969, p. 12) in order to specify gamma distribution functions which approximate $F_2(U)$. Basmann and Richardson (1969b) have derived the following expression for the moments of V_2:

$$(5.1.3)\quad E[V_2^h] = \frac{2^h\,\Gamma\left(\frac{\nu}{2}+h\right)}{\Gamma\left(\frac{\nu}{2}\right)}\, e^{-\frac{\bar{\mu}^2}{2}(1+\bar{\beta}_1^{\,2})}$$

$$\times \sum_{j=0}^{\infty} \frac{\left(\frac{\bar{\mu}^2}{2}\right)^j}{j!}\; {}_1F_1\left[\begin{matrix}\frac{\nu+1}{2}+h+j; \\ \frac{\nu+1}{2}+j;\end{matrix}\; \frac{\bar{\beta}_1^{\,2}\bar{\mu}^2}{2}\right]$$

(Basmann and Richardson, 1969b, p. 15).

The gamma distribution function, G(U), with associated parameter space (a, b) is specified by

$$(5.1.4\text{ a-b})\quad G(U) = \frac{\left(\frac{U}{b}\right)^a}{\Gamma(a+1)}\; {}_1F_1\left[\begin{matrix}a; \\ a+1;\end{matrix}\; -\frac{U}{b}\right]$$

$$0 \leqq U < \infty$$

$$= 0 \quad \text{otherwise}.$$

The moments for a random variable, U, distributed as the gamma distribution are specified by

$$(5.1.5)\qquad E[U^h] = b^h\,\frac{\Gamma(a+h)}{\Gamma(a)}$$

We have then two methods of computing $F_2(U)$, by use of (4.1.1) and by use of (4.1.2) and (5.1.1) together; we have three methods of approximating $F_2(U)$, by use of (4.1.2) and by the two procedures making use of (5.1.3), (5.1.4), and (5.1.5); and by use of (5.1.1) we can compute $F_2(\nu) - \bar{F}_2(\nu)$. Computer programs in the Fortran IV language with double precision real variables have been written suitable to investigate each of the six computational procedures indicated above. Some results of the computations are presented in 6.1.

5.2 Approximations of the Exact Finite Sample Distribution Function Associated with V_3

By repeated application of (A.11) we obtain

$$(5.2.1)\qquad {}_3F_3\left[\begin{matrix}\frac{\nu+2}{2},\ \frac{1}{2}+j,\ \frac{m+\nu+1}{2}+k; \\ \frac{1}{2},\ \frac{\nu+1}{2}+j,\ \frac{m+\nu+2}{2}+k+j;\end{matrix}\ \frac{\mu^{-2}}{2}\right]$$

$$= \sum_{r=0}^{\infty} \frac{\left(-\frac{\nu+1}{2}\right)_r(-j)_r\left(\frac{m+\nu+1}{2}+k\right)_r}{\left(\frac{1}{2}\right)_r\left(\frac{\nu+1}{2}+j\right)_r\left(\frac{m+\nu+2}{2}+k+j\right)_r} \cdot \frac{\left(\frac{\mu^{-2}}{2}\right)^r}{r!}$$

$$\times \sum_{s=0}^{\infty} \frac{\left(-\frac{1}{2}+r\right)_s\left(-\frac{m}{2}-k+j\right)_s}{\left(\frac{\nu+1}{2}+j+r\right)_s\left(\frac{m+\nu+2}{2}+k+j+r\right)_s} \cdot \frac{\left(\frac{\mu^{-2}}{2}\right)^s}{s!}$$

$$\times \; {}_1F_1 \left[\begin{matrix} \frac{m+\nu+2}{2} + k; \\ \frac{m+\nu+2}{2} + k + j + r + s; \end{matrix} \; \frac{\bar{\mu}^2}{2} \right]$$

The asymptotic expansion of the confluent hypergeometric function in (5.2.1) is approximated by

$$(5.2.2) \quad {}_1F_1 \left[\begin{matrix} \frac{m+\nu+2}{2} + k; \\ \frac{m+\nu+2}{2} + k + j + r + s; \end{matrix} \; \frac{\bar{\mu}^2}{2} \right] \simeq$$

$$e^{\frac{\bar{\mu}^2}{2}} \left(\frac{\bar{\mu}^2}{2}\right)^{-j-r-s} \left(\frac{m+\nu+2}{2} + k\right)_{j+r+s}$$

$$\times \sum_{t=0}^{S-1} \frac{(j+r+s)_t \left(- \frac{m+\nu}{2} - k\right)_t}{t!} \left(\frac{\bar{\mu}^2}{2}\right)^{-t}$$

(Slater, 1960, p. 60).

In (4.2) we showed that $F_3(U) \to \bar{F}_3(U)$ as $\bar{\mu}^2 \to \infty$. Then from (4.2.1), (5.2.1), (5.2.2), and (A.2) we have

$$(5.2.3 \text{ a-b}) \quad F_3(U) - \bar{F}_3(U) \simeq \frac{-2}{\bar{\mu}^2} \cdot \frac{\left[\left(1 + \bar{\beta}_1^{\,2}\right)\frac{U}{2}\right]^{\frac{m}{2}}}{\Gamma\left(\frac{m+2}{2}\right)}$$

$$\times \left[\frac{\Gamma\left(\frac{\nu+2}{2}\right)\Gamma\left(\frac{m+\nu+1}{2}\right)}{\Gamma\left(\frac{\nu+1}{2}\right)\Gamma\left(\frac{m+\nu+2}{2}\right)}\right]$$

$$\times \sum_{k=0}^{\infty} \frac{\left(\frac{m}{2}\right)_k \left(\frac{m+\nu+1}{2}\right)_k \left[-\left(1+\bar{\beta}_1^{\,2}\right)\frac{U}{2}\right]^k}{\left(\frac{m+2}{2}\right)_k \left(\frac{m+\nu+2}{2}\right)_k \; k!}$$

$$\times \left(\frac{m+\nu}{2}+k\right) \sum_{j=0}^{\infty} \frac{\left(\frac{1}{2}\right)_j \left(\frac{m}{2}+k\right)_j \left(-\bar{\beta}_1^{\,2}\right)^j}{\left(\frac{\nu+1}{2}\right)_j \; j!}$$

$$\times \sum_{r=0}^{\infty} \frac{\left(-\frac{\nu+1}{2}\right)_r (-j)_r \left(\frac{m+\nu+1}{2}+k\right)_r}{\left(\frac{1}{2}\right)_r \left(\frac{\nu+1}{2}+j\right)_r \; r!}$$

$$\times \sum_{s=0}^{\infty} \frac{\left(-\frac{1}{2}+r\right)_s \left(-\frac{m}{2}-k+j\right)_s}{\left(\frac{\nu+1}{2}+j+r\right)_s \; s!} \cdot (j+r+s)$$

$0 \leqq U < \infty$

$= 0$ otherwise.

By use of (A.1) and (A.5) we have

$$(5.2.4) \quad \sum_{s=0}^{\infty} \frac{\left(-\frac{1}{2}+r\right)_s \left(-\frac{m}{2}-k+j\right)_s}{\left(\frac{\nu+1}{2}+j+r\right)_s \; s!} \cdot (j+r+s)$$

$$= (j+r) \cdot \frac{\Gamma\left(\frac{\nu+1}{2} + j + r\right)\Gamma\left(\frac{m+\nu+2}{2} + k\right)}{\Gamma\left(\frac{\nu+2}{2} + j\right)\Gamma\left(\frac{m+\nu+1}{2} + k + r\right)}$$

$$+ \frac{\left(-\frac{1}{2} + r\right)\left(-\frac{m}{2} - k + j\right)}{\left(\frac{\nu+1}{2} + j + r\right)} \cdot \frac{\Gamma\left(\frac{\nu+3}{2} + j + r\right)\Gamma\left(\frac{m+\nu}{2} + k\right)}{\Gamma\left(\frac{\nu+2}{2} + j\right)\Gamma\left(\frac{m+\nu+1}{2} + k + r\right)}$$

From (5.2.3), (5.2.4), and (A.2) we obtain

$$\text{(5.2.5 a-b)} \quad F_3(U) - \bar{F}_3(U) \simeq \frac{-2}{\bar{\mu}^2} \cdot \frac{\left[\left(1 + \bar{\beta}_1{}^2\right)\frac{U}{2}\right]^{\frac{m}{2}}}{\Gamma\left(\frac{m+2}{2}\right)}$$

$$\times \sum_{k=0}^{\infty} \frac{\left(\frac{m}{2}\right)_k \left[-\left(1 + \bar{\beta}_1{}^2\right)\frac{U}{2}\right]^k}{\left(\frac{m+2}{2}\right)_k \; k!}$$

$$\times \sum_{j=0}^{\infty} \frac{\left(\frac{1}{2}\right)_j \left(\frac{m}{2} + k\right)_j \left(-\bar{\beta}_1{}^2\right)_j}{\left(\frac{\nu+2}{2}\right)_j \; j!}$$

$$\times \sum_{r=0}^{\infty} \frac{\left(-\frac{\nu+1}{2}\right)_r (-j)_r}{\left(\frac{1}{2}\right)_r \; r!}$$

$$\times \left[\left(\frac{m+\nu}{2} + k\right)(j+r) + \left(-\frac{1}{2} + r\right)\left(-\frac{m}{2} - k + j\right)\right]$$

$0 \leqq U < \infty$

$= 0$ otherwise .

By use of (A.1) and (A.8) we have

(5.2.6) $$\sum_{r=0}^{\infty} \frac{\left(-\frac{\nu+1}{2}\right)_r (-j)_r}{\left(\frac{1}{2}\right)_r \quad r!} \left[\left(\frac{m+\nu}{2} + k\right)\right.$$

$$\times \left.(j+r) + \left(-\frac{1}{2} + r\right)\left(-\frac{m}{2} - k + j\right)\right] =$$

$$\left[j\left(\frac{m+\nu}{2} + k\right) - \frac{1}{2}\left(-\frac{m}{2} - k + j\right)\right.$$

$$\left. + \ j\left(\frac{\nu+1}{2}\right)\right] \frac{\left(\frac{\nu+2}{2}\right)_j}{\left(\frac{1}{2}\right)_j}$$

From (5.2.5) and (5.2.6) we obtain

(5.2.7 a-b) $$F_3(U) - \bar{F}_3(U) \simeq \frac{-2}{\bar{\mu}^2} \cdot \frac{\left[\left(1 + \bar{\beta}_1{}^2\right)\frac{U}{2}\right]^{\frac{m}{2}}}{\Gamma\left(\frac{m+2}{2}\right)}$$

$$\times \sum_{k=0}^{\infty} \frac{\left(\frac{m}{2}\right)_k \left[-\left(1 + \bar{\beta}_1{}^2\right)\frac{U}{2}\right]^k}{\left(\frac{m+2}{2}\right)_k \quad k!}$$

$$\times \sum_{j=0}^{\infty} \frac{\left(\frac{m}{2} + k\right)_j \left(- \bar{\beta}_1^{\,2}\right)^j}{j!}$$

$$\times \left[\frac{1}{2}\left(\frac{m}{2} + k\right) + j\left(\nu + \frac{m}{2} + k\right)\right]$$

$$0 \leqq U < \infty$$

$$= 0 \quad \text{otherwise} .$$

Since $F_3(U) - \bar{F}_3(U)$ is an analytic function of $\bar{\beta}_1^{\,2}$ for finite $\bar{\mu}^2$ and $F_3(U)$ and $\bar{F}_3(U)$ are represented by series convergent for all values of $\bar{\beta}_1^{\,2}$ the results which follow are valid for all values of $\bar{\beta}_1^{\,2}$ by the principle of analytic continuation (Churchill, 1960, p. 262). By use of (A.2) and (A.6) we have

$$\text{(5.2.8)} \quad \sum_{j=0}^{\infty} \frac{\left(\frac{m}{2} + k\right)_j \left(- \bar{\beta}_1^{\,2}\right)^j}{j!}$$

$$\times \left[\frac{1}{2}\left(\frac{m}{2} + k\right) + j\left(\nu + \frac{m}{2} + k\right)\right] =$$

$$\left[\frac{1}{2} - \left(\frac{\bar{\beta}_1^{\,2}}{1+\bar{\beta}_1^{\,2}}\right)\left(\nu + \frac{m}{2} + k\right)\right]$$

$$\times \left(\frac{m}{2} + k\right)\left(1 + \bar{\beta}_1^{\,2}\right)^{-\frac{m}{2} - k}$$

From (5.2.7), (5.2.8), and (A.2) we obtain

$$\text{(5.2.9 a-b)} \quad F_3(U) - \bar{F}_3(U) \simeq \frac{-2}{\bar{\mu}^2} \cdot \frac{\left(\frac{U}{2}\right)^{\frac{m}{2}}}{\Gamma\left(\frac{m}{2}\right)}$$

$$\times \sum_{k=0}^{\infty} \frac{\left(-\frac{U}{2}\right)^k}{k!} \left[\frac{1}{2} - \left(\frac{\bar{\beta}_1^{\,2}}{1+\bar{\beta}_1^{\,2}}\right) \left(\nu + \frac{m}{2} + k\right)\right]$$

$$0 \leqq U < \infty$$

$$= 0 \quad \text{otherwise}.$$

From (5.2.9) and (A.2) we obtain

$$\text{(5.2.10 a-b)} \quad F_3(U) - \bar{F}_3(U) \simeq -\frac{\left(\frac{U}{2}\right)^{\frac{m}{2}} e^{-\frac{U}{2}}}{\bar{\mu}^2 \, \Gamma\left(\frac{m}{2}\right)}$$

$$\times \left[1 - \left(\frac{\bar{\beta}_1^{\,2}}{1+\bar{\beta}_1^{\,2}}\right) \left(m + 2\nu - U\right)\right]$$

$$0 \leqq U < \infty$$

$$= 0 \quad \text{otherwise}.$$

From (5.2.10) we obtain

(5.2.11 a-b) $$\frac{d}{dU}\left[F_3(U) - \bar{F}_3(U)\right] \simeq$$

$$\frac{\left(\frac{U}{2}\right)^{\frac{m}{2}-1} e^{-\frac{U}{2}}}{4\bar{\mu}^2\,\Gamma\left(\frac{m}{2}\right)} \left\{ \left(\frac{\bar{\beta}_1^{\,2}}{1+\bar{\beta}_1^{\,2}}\right) U^2 \right.$$

$$+ \left[1 - 2\,(m+\nu+1)\left(\frac{\bar{\beta}_1^{\,2}}{1+\bar{\beta}_1^{\,2}}\right)\right] U$$

$$\left. - m\left[1 - (m + 2\nu)\left(\frac{\bar{\beta}_1^{\,2}}{1+\bar{\beta}_1^{\,2}}\right)\right]\right\}$$

$$0 \leqq U < \infty$$

$$= 0 \quad \text{otherwise}\ .$$

We note that when $\bar{\beta}_1^{\,2} = 0$ we obtain an extreme value which must be a minimum for $U \simeq m$. For $\bar{\beta}_1^{\,2} \neq 0$ the extreme values occur at those values of $U > 0$ which solve the quadratic equation obtained by setting equal to zero the expression inside curled braces in (5.2.11a).

Let S be the set of U's (at most two) which determine the extreme values of $F_3(U) - \bar{F}_3(U)$. We base a decision on approximating $F_3(U)$ by

$\bar{F}_3(U)$ on the value of $\underset{U \in S}{MAX} \mid F_3(U) - \bar{F}_3(U) \mid$; i.e. when $\underset{U \in S}{MAX} \mid F_3(U) - \bar{F}_3(U) \mid$ is sufficiently small (an arbitrary specification) we approximate $F_3(U)$ by $\bar{F}_3(U)$. Since the moments of V_3, $E[V_3{}^h]$, exist only for $h < \frac{\nu+1}{2}$ (Basmann and Richardson, 1969a, p. 11) it is not possible to use the <u>method of moments</u> to approximate $F_3(U)$ except for $\nu \geqq 4$.

We can then use (4.2.1) to compute $F_3(U)$ and we can use (4.2.2) either alone or with (5.2.10) to approximate $F_3(U)$; $\underset{U \in S}{MAX} \mid F_3(U) - \bar{F}_3(U) \mid$ can be computed by (4.2.1) and (4.2.2) or approximated by (5.2.10). Computer programs in the Fortran IV language with double precision real variables have been written suitable to investigate the computational procedures indicated above. Some results of the computations are presented in 6.2.

<u>5.3 Approximations of the Exact Finite Sample Distribution Function Associated with V_4</u>

By repeated application of (A.11) we obtain

(5.3.1)
$$ {}_3F_3 \left[\begin{matrix} \frac{\nu+2}{2} + k, & \frac{m+\nu+1}{2}, & \frac{1}{2} + q; \\ \frac{1}{2}, & \frac{m+\nu+2}{2} + k + q, & \frac{\nu+1}{2} + s + q; \end{matrix} \; \frac{-\mu^2}{2} \right] $$

$$ = \sum_{j=0}^{\infty} \frac{\left(-\frac{m+\nu}{2}\right)_j (-q)_j \left(\frac{\nu+2}{2} + k\right)_j}{\left(\frac{1}{2}\right)_j \left(\frac{m+\nu+2}{2} + k + q\right)_j \left(\frac{\nu+1}{2} + s + q\right)_j} \cdot \frac{\left(\frac{-\mu^2}{2}\right)^j}{j!} $$

$$\times \sum_{p=0}^{\infty} \frac{\left(-\frac{m}{2}+s+j\right)_p\left(-\frac{1}{2}-k+s+q\right)_p}{\left(\frac{m+\nu+2}{2}+k+q+j\right)_p\left(\frac{\nu+1}{2}+s+q+j\right)_p} \cdot \frac{\left(\frac{\bar{\mu}^2}{2}\right)^p}{p!}$$

$$\times \; {}_1F_1\left[\begin{matrix}\frac{m+\nu+2}{2}+k-s; \\ \frac{m+\nu+2}{2}+k+q+j+p;\end{matrix} \; \frac{\bar{\mu}^2}{2}\right]$$

The asymptotic expansion of the confluent hypergeometric function in (5.3.1) is approximated by

$$(5.3.2) \quad {}_1F_1\left[\begin{matrix}\frac{m+\nu+2}{2}+k-s; \\ \frac{m+\nu+2}{2}+k+q+j+p;\end{matrix} \; \frac{\bar{\mu}^2}{2}\right] \simeq$$

$$e^{\frac{\bar{\mu}^2}{2}} \left(\frac{\bar{\mu}^2}{2}\right)^{-s-q-j-p} \frac{\Gamma\left(\frac{m+\nu+2}{2}+k+q+j+p\right)}{\Gamma\left(\frac{m+\nu+2}{2}+k-s\right)}$$

$$\times \sum_{t=0}^{S-1} \frac{(s+q+j+p)_t\left(-\frac{m+\nu}{2}-k+s\right)_t}{t!} \left(\frac{\bar{\mu}^2}{2}\right)^{-t}$$

(Slater, 1960, p. 60).

In (4.3) we showed that $F_4(U) \to \bar{F}_4(U)$ as $\bar{\mu}^2 \to \infty$. Then from (4.3.1), (5.3.1), (5.3.2), and (A.2) we have

$$(5.3.3\text{ a-b})\quad F_4(U) - \bar{F}_4(U) \simeq \frac{-2}{\bar{\mu}^2} \cdot \frac{\left[\left(1+\bar{\beta}_1^{\,2}\right)\frac{U}{2}\right]^{\frac{m+\nu}{2}}}{\Gamma\left(\frac{m+\nu+2}{2}\right)}$$

$$\times \left[\frac{\Gamma\left(\frac{\nu+2}{2}\right)\Gamma\left(\frac{m+\nu+1}{2}\right)}{\Gamma\left(\frac{\nu+1}{2}\right)} \right]$$

$$\times \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n\left(\frac{m+\nu}{2}\right)_{n+r}}{\left(\frac{m+\nu}{2}\right)_n\left(\frac{m+\nu+2}{2}\right)_{n+r}} \cdot \frac{(-)^r\left[\left(1+\bar{\beta}_1^{\,2}\right)\frac{U}{2}\right]^{n+r}}{n!\quad r!}$$

$$\times \sum_{k=0}^{\infty} \frac{(-n)_k\left(\frac{\nu+2}{2}\right)_k}{k!}$$

$$\times \sum_{s=0}^{\infty}\sum_{q=0}^{\infty} \frac{\left(\frac{m}{2}\right)_q\left(\frac{1}{2}+k\right)_q\left(\frac{\nu}{2}\right)_s\left(\frac{m+\nu}{2}+n+r\right)_s}{\left(\frac{m+\nu}{2}+n\right)_s\left(\frac{\nu+1}{2}\right)_{s+q}} \cdot \frac{\left(-\bar{\beta}_1^{\,2}\right)^{s+q}}{s!\quad q!}$$

$$\times \sum_{j=0}^{\infty} \frac{\left(-\frac{m+\nu}{2}\right)_j(-q)_j\left(\frac{\nu+2}{2}+k\right)_j}{\left(\frac{1}{2}\right)_j\left(\frac{\nu+1}{2}+s+q\right)_j\quad j!}$$

$$\times \sum_{p=0}^{\infty} \frac{\left(-\frac{m}{2}+s+j\right)_p\left(-\frac{1}{2}-k+s+q\right)_p}{\left(\frac{\nu+1}{2}+s+q+j\right)_p\quad p!}$$

$$\times \quad \frac{(s + q + j + p)\left(\frac{m+\nu}{2} + k - s\right)}{\Gamma\left(\frac{m+\nu+2}{2} + k - s\right)}$$

$$0 \leqq U < \infty$$

$$= 0 \quad \text{otherwise.}$$

By use of (A.1) and (A.5) we have

$$(5.3.4) \quad \sum_{p=0}^{\infty} \frac{\left(-\frac{m}{2} + s + j\right)_p \left(-\frac{1}{2} - k + s + q\right)_p}{\left(\frac{\nu+1}{2} + s + q + j\right)_p \quad p!}$$

$$\times \quad \frac{(s + q + j + p)\left(\frac{m+\nu}{2} + k - s\right)}{\Gamma\left(\frac{m+\nu+2}{2} + k - s\right)}$$

$$= \quad \frac{(s + q + j)\left(\frac{m+\nu}{2} + k - s\right)\Gamma\left(\frac{\nu+1}{2} + s + q + j\right)}{\Gamma\left(\frac{m+\nu+1}{2} + q\right)\Gamma\left(\frac{\nu+2}{2} + k + j\right)}$$

$$+ \quad \frac{\left(\frac{m+\nu}{2} + k - s\right)\left(-\frac{m}{2} + s + j\right)\left(-\frac{1}{2} - k + s + q\right)}{\Gamma\left(\frac{m+\nu+2}{2} + k - s\right)\left(\frac{\nu+1}{2} + s + q + j\right)}$$

$$\times \quad \frac{\Gamma\left(\frac{\nu+3}{2} + s + q + j\right)\Gamma\left(\frac{m+\nu}{2} + k - s\right)}{\Gamma\left(\frac{m+\nu+1}{2} + q\right)\Gamma\left(\frac{\nu+2}{2} + k + j\right)}$$

From (5.3.3), (5.3.4), and (A.2) we obtain

$$\text{(5.3.5 a-b)}\quad F_4(U) - \overline{F}_4(U) \simeq \frac{-2}{\overline{\mu}^{2}} \cdot \frac{\left[\left(1+\overline{\beta}_1^{\,2}\right)\frac{U}{2}\right]^{\frac{m+\nu}{2}}}{\Gamma\left(\frac{m+\nu+2}{2}\right)}$$

$$\times \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n\left(\frac{m+\nu}{2}\right)_{n+r}(-n)_k}{\left(\frac{m+\nu}{2}\right)_n\left(\frac{m+\nu+2}{2}\right)_{n+r}} \cdot \frac{(-)^r\left[\left(1+\overline{\beta}_1^{\,2}\right)\frac{U}{2}\right]^{n+r}}{n!\quad r!\qquad k!}$$

$$\times \sum_{s=0}^{\infty}\sum_{q=0}^{\infty} \frac{\left(\frac{m}{2}\right)_q\left(\frac{1}{2}+k\right)_q\left(\frac{\nu}{2}\right)_s\left(\frac{m+\nu}{2}+n+r\right)_s}{\left(\frac{m+\nu}{2}+n\right)_s\left(\frac{m+\nu+1}{2}\right)_q} \cdot \frac{\left(-\overline{\beta}_1^{\,2}\right)^{s+q}}{s!\quad q!}$$

$$\times \sum_{j=0}^{\infty} \frac{\left(-\frac{m+\nu}{2}\right)_j(-q)_j}{\left(\frac{1}{2}\right)_j\quad j!} \left[(s+q)\left(\frac{m+\nu}{2}+k-s\right)\right.$$

$$\left. + \left(-\frac{m}{2}+s\right)\left(-\frac{1}{2}-k+s+q\right) + j\left(\frac{m+\nu-1}{2}+q\right)\right]$$

$$0 \leqq U < \infty$$

$$= 0 \quad \text{otherwise.}$$

By use of (A.1) and (A.8) we have

$$(5.3.6)\quad \sum_{j=0}^{\infty} \frac{\left(-\frac{m+\nu}{2}\right)_j (-q)_j}{\left(\frac{1}{2}\right)_j \; j!} \left[(s+q)\left(\frac{m+\nu}{2} + k - s\right)\right.$$

$$\left. + \left(-\frac{m}{2} + s\right)\left(-\frac{1}{2} - k + s + q\right) + j\left(\frac{m+\nu-1}{2} + q\right)\right] =$$

$$\frac{\left(\frac{m+\nu+1}{2}\right)_q}{\left(\frac{1}{2}\right)_q} \left[(s+q)\left(\frac{\nu}{2}\right) + q\left(\frac{m+\nu}{2}\right) + \frac{m}{4} - \frac{s}{2} + k\left(q + \frac{m}{2}\right)\right]$$

From (5.3.5) and (5.3.6) we obtain

$$(5.3.7\ \text{a-b})\quad F_4(U) - \overline{F}_4(U) \simeq \frac{-2}{\overline{\mu}^2} \cdot \frac{\left[\left(1 + \overline{\beta}_1^{\,2}\right)\frac{U}{2}\right]^{\frac{m+\nu}{2}}}{\Gamma\left(\frac{m+\nu+2}{2}\right)}$$

$$\times \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{m+\nu}{2}\right)_{n+r}}{\left(\frac{m+\nu}{2}\right)_n \left(\frac{m+\nu+2}{2}\right)_{n+r}} \cdot \frac{(-)^r \left[\left(1 + \overline{\beta}_1^{\,2}\right)\frac{U}{2}\right]^{n+r}}{n! \quad r!}$$

$$\times \sum_{s=0}^{\infty} \sum_{q=0}^{\infty} \frac{\left(\frac{m}{2}\right)_q \left(\frac{\nu}{2}\right)_s \left(\frac{m+\nu}{2} + n + r\right)_s}{\left(\frac{m+\nu}{2} + n\right)_s} \cdot \frac{\left(-\overline{\beta}_1^{\,2}\right)^{s+q}}{s! \quad q!}$$

$$\times \sum_{k=0}^{\infty} \frac{(-n)_k \left(\frac{1}{2} + q\right)_k}{\left(\frac{1}{2}\right)_k k!} \left[(s+q)\left(\frac{\nu}{2}\right) + q\left(\frac{m+\nu}{2}\right) \right.$$

$$\left. + \frac{m}{4} - \frac{s}{2} + k\left(q + \frac{m}{2}\right) \right]$$

$$0 \leqq U < \infty$$

$$= 0 \quad \text{otherwise.}$$

By use of (A.1) and (A.8) we have

$$(5.3.8) \quad \sum_{k=0}^{\infty} \frac{(-n)_k \left(\frac{1}{2} + q\right)_k}{\left(\frac{1}{2}\right)_k k!} \left[(s+q)\left(\frac{\nu}{2}\right) + q\left(\frac{m+\nu}{2}\right) + \frac{m}{4} - \frac{s}{2} + k\left(q + \frac{m}{2}\right)\right] =$$

$$\frac{(-q)_n}{\left(\frac{1}{2}\right)_n} \left[(s+q)\left(\frac{\nu}{2}\right) + q\left(\frac{m+\nu}{2}\right) + \frac{m}{4} - \frac{s}{2}\right]$$

$$+ \frac{(-q)_{n-1}}{\left(\frac{1}{2}\right)_n} \left(q + \frac{m}{2}\right) (-n) \left(\frac{1}{2} + q\right)$$

From (5.3.7) and (5.3.8), replacing $\underline{r}$ by $\underline{r-n}$ and making use of (A.7) we obtain

$$(5.3.9\text{ a-b})\qquad F_4(U) - \bar{F}_4(U) \simeq \frac{-2}{\bar{\mu}^2} \cdot \frac{\left[\left(1+\bar{\beta}_1^{\,2}\right)\frac{U}{2}\right]^{\frac{m+\nu}{2}}}{\Gamma\left(\frac{m+\nu+2}{2}\right)}$$

$$\times \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \frac{\left(\frac{m+\nu}{2}\right)_r (-r)_n}{\left(\frac{m+\nu}{2}\right)_n \left(\frac{m+\nu+2}{2}\right)_r} \cdot \frac{\left[-\left(1+\bar{\beta}_1^{\,2}\right)\frac{U}{2}\right]^r}{n!\quad r!}$$

$$\times \sum_{s=0}^{\infty}\sum_{q=0}^{\infty} \frac{\left(\frac{m}{2}\right)_q \left(\frac{\nu}{2}\right)_s \left(\frac{m+\nu}{2}+r\right)_s}{\left(\frac{m+\nu}{2}+n\right)_s} \cdot \frac{\left(-\bar{\beta}_1^{\,2}\right)^{s+q}}{s!\quad q!}$$

$$\times \left\{ \left[(s+q)\left(\frac{\nu}{2}\right) + q\left(\frac{m+\nu}{2}\right) + \frac{m}{4} - \frac{s}{2}\right](-q)_n\right.$$

$$\left. + \left(q + \frac{m}{2}\right)(-n)\left(\frac{1}{2}+q\right)(-q)_{n-1}\right\}$$

$$0 \leqq U < \infty$$

$$= 0 \quad \text{otherwise}.$$

By use of (A.1) and (A.8) we have

$$(5.3.10)\qquad \sum_{n=0}^{\infty} \frac{(-r)_n}{\left(\frac{m+\nu}{2}+s\right)_n n!} \left\{ \left[(s+q)\left(\frac{\nu}{2}\right)\right.\right.$$

$$+ \quad q\left(\frac{m+\nu}{2}\right) + \frac{m}{4} - \frac{s}{2} \Big] (-q)_n + \left(q + \frac{m}{2}\right)$$

$$\times \quad (-n) \left(\frac{1}{2} + q\right)(-q)_{n-1} \Big\} =$$

$$\frac{\left(\frac{m+\nu}{2} + s + r\right)_q}{\left(\frac{m+\nu}{2} + s\right)_q} \left[(s+q)\left(\frac{\nu}{2}\right) + q\left(\frac{m+\nu}{2}\right) + \frac{m}{4} - \frac{s}{2} \right]$$

$$+ \frac{\left(\frac{m+\nu}{2} + s + r\right)_q}{\left(\frac{m+\nu}{2} + s\right)_{q+1}} \left(q + \frac{m}{2}\right)\left(\frac{1}{2} + q\right) (r)$$

From (5.3.9) and (5.3.10), by use of (A.2) we obtain

$$\text{(5.3.11 a-b)} \quad F_4(U) - \bar{F}_4(U) \simeq \frac{-2}{\bar{\mu}^{2}} \cdot \frac{\left[\left(1 + \bar{\beta}_1{}^2\right)\frac{U}{2}\right]^{\frac{m+\nu}{2}}}{\Gamma\left(\frac{m+\nu+2}{2}\right)}$$

$$\times \sum_{r=0}^{\infty} \frac{\left(\frac{m+\nu}{2}\right)_r}{\left(\frac{m+\nu+2}{2}\right)_r} \cdot \frac{\left[-\left(1 + \bar{\beta}_1{}^2\right)\frac{U}{2}\right]^r}{r!}$$

$$\times \left\{ \sum_{s=0}^{\infty} \sum_{q=0}^{\infty} \frac{\left(\frac{m}{2}\right)_q \left(\frac{\nu}{2}\right)_s \left(\frac{m+\nu}{2} + r\right)_{s+q}}{\left(\frac{m+\nu}{2}\right)_{s+q}} \cdot \frac{\left(-\bar{\beta}_1{}^2\right)^{s+q}}{s! \quad q!}\right.$$

$$\times \left[(s+q)\left(\frac{\nu}{2}\right) + q\left(\frac{m+\nu}{2}\right) + \frac{m}{4} - \frac{s}{2} \right]$$

$$+ \sum_{s=0}^{\infty} \sum_{q=0}^{\infty} \frac{\left(\frac{m}{2}\right)_q \left(\frac{\nu}{2}\right)_s \left(\frac{m+\nu}{2} + r\right)_{s+q}}{\left(\frac{m+\nu}{2}\right)_{s+q+1}} \cdot \frac{\left(-\bar{\beta}_1^{\,2}\right)^{s+q}}{s! \quad q!}$$

$$\times \ (r) \left(q + \frac{m}{2}\right)\left(\frac{1}{2} + q\right) \Big\}$$

$$0 \leqq U < \infty$$

$$= 0 \quad \text{otherwise}.$$

By use of (A.1), (A.6), (A.7), and (A.8) we have

$$(5.3.12) \quad \sum_{s=0}^{\infty} \frac{\left(\frac{\nu}{2}\right)_s \left(\frac{m+\nu}{2} + r\right)_s}{\left(\frac{m+\nu}{2}\right)_s} \cdot \frac{\left(-\bar{\beta}_1^{\,2}\right)^s}{s!}$$

$$\times \sum_{q=0}^{\infty} \frac{\left(\frac{m}{2}\right)_q \left(-s\right)_q}{\left(1-\frac{\nu}{2} - s\right)_q \quad q!} \left[s\left(\frac{\nu-1}{2}\right) + q\left(\frac{m+\nu+1}{2}\right) + \frac{m}{4} \right]$$

$$= (1 + \bar{\beta}_1^{\,2})^{-\frac{m+\nu}{2} - r} \left[\frac{m}{4} + \left(\frac{\nu-1}{2}\right)\left(\frac{-\bar{\beta}_1^{\,2}}{1+\bar{\beta}_1^{\,2}}\right) \left(\frac{m+\nu}{2} + r\right) \right.$$

$$+ \frac{m}{2} \left(\frac{m+\nu+1}{2}\right) \left(\frac{-\bar{\beta}_1^{\,2}}{1+\bar{\beta}_1^{\,2}}\right)\left(\frac{m+\nu}{2} + r\right)/\left(\frac{m+\nu}{2}\right)\Big]$$

and making use of (A.14) also, we have

$$(5.3.13) \qquad r \sum_{s=0}^{\infty} \frac{\left(\frac{\nu}{2}\right)_s \left(\frac{m+\nu}{2} + r\right)_s}{\left(\frac{m+\nu}{2}\right)_{s+1}} \cdot \frac{\left(-\bar{\beta}_1^{\,2}\right)^s}{s!}$$

$$\times \sum_{q=0}^{\infty} \frac{\left(\frac{m}{2}\right)_q (-s)_q}{\left(1 - \frac{\nu}{2} - s\right)_q \; q!} \left(q + \frac{m}{2}\right) \left(\frac{1}{2} + q\right) =$$

$$\frac{r\left(1 + \bar{\beta}_1^{\,2}\right)^{-\frac{m+\nu}{2} - r}}{\left(\frac{m+\nu}{2}\right)} \Big[\frac{m}{4} \sum_{s=0}^{\infty} \frac{\left(\frac{m+\nu}{2} + r\right)_s}{\left(\frac{m+\nu+2}{2}\right)_s} \cdot \left(\frac{\bar{\beta}_1^{\,2}}{1+\bar{\beta}_1^{\,2}}\right)^s$$

$$- \frac{m}{2} \left(\frac{m+3}{2}\right) \left(\frac{\bar{\beta}_1^{\,2}}{1+\bar{\beta}_1^{\,2}}\right) \cdot \frac{\left(\frac{m+\nu}{2}\right)\left(\frac{m+\nu+2}{2}\right)_r}{\left(\frac{m+\nu+2}{2}\right)\left(\frac{m+\nu}{2}\right)_r}$$

$$\times \sum_{s=0}^{\infty} \frac{\left(\frac{m+\nu+2}{2} + r\right)_s}{\left(\frac{m+\nu+4}{2}\right)_s} \cdot \left(\frac{\bar{\beta}_1^{\,2}}{1+\bar{\beta}_1^{\,2}}\right)^s$$

$$+ \frac{m}{2}\left(\frac{m+2}{2}\right)\left(\frac{\bar{\beta}_1^{\,2}}{1+\bar{\beta}_1^{\,2}}\right)^2 \cdot \frac{\left(\frac{m+\nu}{2}\right)\left(\frac{m+\nu+4}{2}\right)_r}{\left(\frac{m+\nu+4}{2}\right)\left(\frac{m+\nu}{2}\right)_r}$$

$$\times \sum_{s=0}^{\infty} \frac{\left(\frac{m+\nu+4}{2}+r\right)_s}{\left(\frac{m+\nu+6}{2}\right)_s} \cdot \left(\frac{\bar{\beta}_1^{\,2}}{1+\bar{\beta}_1^{\,2}}\right)^s \Big]$$

Since $F_4(U) - \bar{F}_4(U)$ is an analytic function of $\bar{\beta}_1^{\,2}$ for finite $\bar{\mu}^2$ and $F_4(U)$ and $\bar{F}_4(U)$ are represented by series convergent for all values of $\bar{\beta}_1^{\,2}$ the results which follow from (5.3.12) and (5.3.13) are valid for all values of $\bar{\beta}_1^{\,2}$ by the principle of analytic continuation (Churchill, 1960, p. 262). Replacing _s_ by _s-q_ in (5.3.11a) and making use of (A.1), (A.7), (5.3.12), and (5.3.13) we obtain

(5.3.14 a-b) $$F_4(U) - \bar{F}_4(U) \simeq \frac{-2\left(\frac{U}{2}\right)^{\frac{m+\nu}{2}}}{\bar{\mu}^2\, \Gamma\left(\frac{m+\nu+2}{2}\right)}$$

$$\times \Big[\frac{m}{4} \sum_{r=0}^{\infty} \frac{\left(\frac{m+\nu}{2}\right)_r}{\left(\frac{m+\nu+2}{2}\right)_r} \cdot \frac{\left(-\frac{U}{2}\right)^r}{r!}$$

$$+ \left(\frac{\nu-1}{2}\right)\left(\frac{-\bar{\beta}_1^{\,2}}{1+\bar{\beta}_1^{\,2}}\right)\left(\frac{m+\nu}{2}\right) e^{-\frac{U}{2}}$$

$$- \ \frac{m}{2}\left(\frac{m+\nu+1}{2}\right)\left(\frac{\bar{\beta}_1^{\,2}}{1+\bar{\beta}_1^{\,2}}\right) e^{-\frac{U}{2}}$$

$$+ \ \frac{\left(\frac{m}{4}\right)}{\left(\frac{m+\nu}{2}\right)} \sum_{r=0}^{\infty} \frac{\left(\frac{m+\nu}{2}\right)_r}{\left(\frac{m+\nu+2}{2}\right)_r} \ \cdot \ \frac{\left(-\frac{U}{2}\right)^r}{r!} \ \cdot \ r$$

$$+ \ \frac{\left(\frac{m}{4}\right)}{\left(\frac{m+\nu+2}{2}\right)} \cdot \left(\frac{\bar{\beta}_1^{\,2}}{1+\bar{\beta}_1^{\,2}}\right) \sum_{r=0}^{\infty} \frac{\left(-\frac{U}{2}\right)^r}{r!} \ \cdot \ r$$

$$\times \ \sum_{s=0}^{\infty} \frac{\left(\frac{m+\nu+2}{2}+r\right)_s}{\left(\frac{m+\nu+4}{2}\right)_s} \left(\frac{\bar{\beta}_1^{\,2}}{1+\bar{\beta}_1^{\,2}}\right)^s$$

$$- \ \frac{\left(\frac{m}{2}\right)\left(\frac{m+3}{2}\right)\left(\frac{\bar{\beta}_1^{\,2}}{1+\bar{\beta}_1^{\,2}}\right)}{\left(\frac{m+\nu+2}{2}\right)} \sum_{r=0}^{\infty} \frac{\left(-\frac{U}{2}\right)^r}{r!} \ \cdot \ r$$

$$\times \ \sum_{s=0}^{\infty} \frac{\left(\frac{m+\nu+2}{2}+r\right)_s}{\left(\frac{m+\nu+4}{2}\right)_s} \left(\frac{\bar{\beta}_1^{\,2}}{1+\bar{\beta}_1^{\,2}}\right)^s$$

$$+ \ \frac{\left(\frac{m}{2}\right)\left(\frac{m+2}{2}\right)\left(\frac{\bar{\beta}_1^{\,2}}{1+\bar{\beta}_1^{\,2}}\right)^2}{\left(\frac{m+\nu+4}{2}\right)} \sum_{r=0}^{\infty} \frac{\left(\frac{m+\nu+4}{2}\right)_r}{\left(\frac{m+\nu+2}{2}\right)_r} \ \cdot \ \frac{\left(-\frac{U}{2}\right)^r}{r!}$$

$$\times \; r \sum_{s=0}^{\infty} \frac{\left(\frac{m+\nu+4}{2} + r\right)_s}{\left(\frac{m+\nu+6}{2}\right)_s} \left(\frac{\bar{\beta}_1^{\,2}}{1+\bar{\beta}_1^{\,2}}\right)^s \Bigg]$$

$$0 \leqq U < \infty$$

$$= 0 \quad \text{otherwise} .$$

From (5.3.14) by use of (A.4) and (A.8) we obtain

(5.3.15 a-b)
$$F_4(U) - \bar{F}_4(U) \simeq \frac{-2\left(\frac{U}{2}\right)^{\frac{m+\nu}{2}}}{\mu^2\,\Gamma\left(\frac{m+\nu+2}{2}\right)}$$

$$\times \left\{\frac{m}{4}\, e^{-\frac{U}{2}} + \frac{1}{4}\left(\frac{\bar{\beta}_1^{\,2}}{1+\bar{\beta}_1^{\,2}}\right) e^{-\frac{U}{2}} \left[\nu - (m+\nu)^2\right]\right.$$

$$- \frac{\left(\frac{m}{2}\right)\left(\frac{m+2}{2}\right)\left(-\frac{U}{2}\right)\left(\frac{\bar{\beta}_1^{\,2}}{1+\bar{\beta}_1^{\,2}}\right)}{\left(\frac{m+\nu+2}{2}\right)} \sum_{s=0}^{\infty} \left(\frac{\bar{\beta}_1^{\,2}}{1+\bar{\beta}_1^{\,2}}\right)^s$$

$$\times \; {}_1F_1\left[\begin{matrix} \frac{m+\nu+4}{2} + s; & \\ \frac{m+\nu+4}{2}\;; & -\frac{U}{2} \end{matrix}\right]$$

$$+ \frac{\left(\frac{m}{2}\right)\left(\frac{m+2}{2}\right)\left(-\frac{U}{2}\right)\left(\frac{\bar{\beta}_1^{\,2}}{1+\bar{\beta}_1^{\,2}}\right)^2}{\left(\frac{m+\nu+2}{2}\right)} \sum_{s=0}^{\infty} \left(\frac{\bar{\beta}_1^{\,2}}{1+\bar{\beta}_1^{\,2}}\right)^s$$

$$\times \quad {}_1F_1\left[\begin{matrix}\frac{m+\nu+6}{2}+s\ ; & \\ & -\frac{U}{2} \\ \frac{m+\nu+4}{2}\ ; & \end{matrix}\right] \quad \Bigg\}$$

$$0 \leqq U < \infty$$

$$= 0 \quad \text{otherwise} .$$

We have

$$(5.3.16) \quad \sum_{s=0}^{\infty} \left(\frac{\bar{\beta}_1^{\,2}}{1+\bar{\beta}_1^{\,2}}\right)^s \quad {}_1F_1\left[\begin{matrix}\frac{m+\nu+4}{2}+s\ ; & \\ & -\frac{U}{2} \\ \frac{m+\nu+4}{2}\ ; & \end{matrix}\right]$$

$$= e^{-\frac{U}{2}} + \left(\frac{\bar{\beta}_1^{\,2}}{1+\bar{\beta}_1^{\,2}}\right) \sum_{s=0}^{\infty} \left(\frac{\bar{\beta}_1^{\,2}}{1+\bar{\beta}_1^{\,2}}\right)^s$$

$$\times \quad {}_1F_1\left[\begin{matrix}\frac{m+\nu+6}{2}+s\ ; & \\ & -\frac{U}{2} \\ \frac{m+\nu+4}{2}\ ; & \end{matrix}\right]$$

From (5.3.15) and (5.3.16) we obtain

$$(5.3.17\ \text{a-b}) \quad F_4(U) - \bar{F}_4(U) \simeq \frac{-\left(\frac{U}{2}\right)^{\frac{m+\nu}{2}} e^{-\frac{U}{2}}}{2\bar{\mu}^2\ \Gamma\left(\frac{m+\nu+2}{2}\right)}$$

$$\times \left\{ m + \left(\frac{\bar{\beta}_1^{\,2}}{1+\bar{\beta}_1^{\,2}}\right) \left[\nu - (m+\nu)^2 \right. \right.$$

$$\left.\left. + \frac{m(m+2)}{(m+\nu+2)} \cdot U \right] \right\}$$

$$0 \leqq U < \infty$$

$$= 0 \quad \text{otherwise} .$$

From (5.3.17) we obtain

(5.3.18 a-b) $\frac{d}{dU} \left[F_4(U) - \bar{F}_4(U) \right] \simeq$

$$\frac{\left(\frac{U}{2}\right)^{\frac{m+\nu}{2} - 1} e^{-\frac{U}{2}}}{8\bar{\mu}^2 \, \Gamma\left(\frac{m+\nu+2}{2}\right)} \left\{ U^2 \left(\frac{\bar{\beta}_1^{\,2}}{1+\bar{\beta}_1^{\,2}}\right) \left[\frac{m(m+2)}{m+\nu+2}\right] \right.$$

$$+ U \left[m + \left(\frac{\bar{\beta}_1^{\,2}}{1+\bar{\beta}_1^{\,2}}\right) \left[\nu - (m+\nu)^2 \right. \right.$$

$$\left. \left. - m\,(m+2) \right] \right] - (m+\nu) \left[m + \left(\frac{\bar{\beta}_1^{\,2}}{1+\bar{\beta}_1^{\,2}}\right) \right.$$

$$\left. \left. \times \left[\nu - (m+\nu)^2 \right] \right] \right\}$$

$$0 \leqq U < \infty$$

$$= 0 \quad \text{otherwise} .$$

We note that when $\bar{\beta}_1^{\,2} = 0$ we obtain an extreme value which must be a minimum for $U \simeq m+\nu$. For $\bar{\beta}_1^{\,2} \neq 0$ the extreme values occur at those values of $U > 0$ which solve the quadratic equation obtained by setting equal to zero the expression inside curled braces in (5.3.18a).

Let S be the set of U's (at most two) which determine the extreme values of $F_4(U) - \bar{F}_4(U)$. We base a decision on approximating $F_4(U)$ by $\bar{F}_4(U)$ on the value of $\underset{U \in S}{\text{MAX}} \mid F_4(U) - \bar{F}_4(U) \mid$; i.e. when $\underset{U \in S}{\text{MAX}} \mid F_4(U) - \bar{F}_4(U) \mid$ is sufficiently small (an arbitrary specification) we approximate $F_4(U)$ by $\bar{F}_4(U)$. Since the moments of V_4, $E[V_4^{\,h}]$, exist only for $h < \frac{\nu+1}{2}$ (Basmann, Ebbeler, and Richardson, 1970, p. 14) it is not possible to use the method of moments to approximate $F_4(U)$ except for $\nu \geqq 4$.

We can then use (4.3.1) to compute $F_4(U)$ and we can use (4.3.2) either alone or with (5.3.17) to approximate $F_4(U)$; $\underset{U \in S}{\text{MAX}} \mid F_4(U) - \bar{F}_4(U) \mid$ can be computed by (4.3.1) and (4.3.2) or approximated by (5.3.17). Computer programs in the Fortran IV language with double precision real variables have been written suitable to investigate the computational procedures indicated above. Some results of the computations are presented in 6.3.

CHAPTER 6

TABULATIONS OF THE DISTRIBUTION FUNCTIONS FOR THE GCL STRUCTURAL VARIANCE ESTIMATORS

6.1 Tabulations of the Exact Finite Sample Distribution Function Associated with V_2

In this section we present tabulations of $\bar{F}_2(U)$, $F_2(U)$, and $F_2(U)-\bar{F}_2(U)$ for $\nu = 1, 2, 3, 4, 5$, $\bar{\beta}_1^{\,2} = .25, 1$, and $\bar{\mu}^2 = 10$ where the tabulations are carried out according to (4.1.2) and (5.1.1) (Tables 6.1.1 - 6.1.5). Notice that for $\nu = 1$, by (5.1.1) $F_2(U) \approx \bar{F}_2(U)$. We tabulate $F_2(U)$ and $F_2(U) - \bar{F}_2(U)$ for $\nu = 1$, $\bar{\beta}_1^{\,2} = .25$, and $\bar{\mu}^2 = 10$ according to (4.1.1) and (4.1.2) (Table 6.1.6) for purpose of comparison with the results obtained by the aforementioned method, $F_2(U) \approx \bar{F}_2(U)$. The computations for $\nu = 2, 3, 4, 5$ have also been made and are comparable in accuracy to those reported here for $\nu = 1$. For all values of ν, $\bar{\beta}_1^{\,2}$, $\bar{\mu}^2$, and U which were investigated the computation employing (5.1.1) was significantly faster than that making use of (4.1.1).

From (5.1.2) we obtained that the maximum value of $F_2(U) - \bar{F}_2(U)$ occurs for $U \approx \nu$. In Table 6.1.7 we present tabulations of $F_2(\nu) - \bar{F}_2(\nu)$ for $\nu = 2, 3, 4, 5$, $\bar{\beta}_1^{\,2} = .25, 1$, and $\bar{\mu}^2 = 10, 20, \ldots, 90, 100, 150, 200, \ldots, 450, 500$ according to (5.1.1).

In Table 6.1.8 we present tabulations of $E[V_2]$ and $E[V_2^{\,2}]$, $(\bar{a},\bar{b})$, and $(\bar{\bar{a}}, \bar{\bar{b}})$ for $\nu = 1, 2, 3, 4, 5$, $\bar{\beta}_1^{\,2} = .25, 1$, and $\bar{\mu}^2 = 10$.

$(\bar{a}, \bar{b})$ are values of the parameters of the approximating gamma distribution function as determined by the method of moments (Kendall and Stuart, 1963, pp. 148-152) and $(\bar{\bar{a}}, \bar{\bar{b}})$ are values as determined by the modified method of moments (Ebbeler and McDonald, 1969, p. 12). If we replace U by 2U/b and ν by 2a in (4.1.2) we obtain the functional form defined by (5.1.4). Therefore (4.1.2) and (5.1.1) can be employed to compute G(U) and $F_2(U) - G(U)$ by the two methods of moments discussed (Tables 6.1.9 - 6.1.12). We omit these tabulations for $\nu = 1$ since we have shown that $F_2(U) \approx \bar{F}_2(U)$ when $\nu = 1$ regardless of other parameter values.

If, in a given application, we have specified that we use $\bar{F}_2(U)$ to approximate $F_2(U)$ only if $F_2(U) - \bar{F}_2(U) < \varepsilon$, $\varepsilon > 0$, and having estimated $\bar{\beta}_1^{\,2}$ by $\hat{\beta}_1^{\,2}$ and $\bar{\mu}^2$ by $\hat{\mu}^2$ we obtain that $F_2(\nu) - \bar{F}_2(\nu) \geq \varepsilon$, we have then several alternative methods of approximating $F_2(U)$. We may compute $F_2(U)$ for ν, $\hat{\beta}_1^{\,2}$, $\hat{\mu}^2$ and we may use either method of moments to approximate $F_2(U)$ for ν, $\hat{\beta}_1^{\,2}$, $\hat{\mu}^2$. Whether we use a computed $F_2(U)$, a gamma distribution function approximation, or use $\bar{F}_2(U)$ as an approximation, the selected distribution function is then employed in tests of hypotheses involving ω_{11}.

For more extensive tables see the article by Ebbeler and McDonald, 1969.

TABLE 6.1.1

$\overline{F}_2(U)$: $\nu = 1$

U	$\overline{F}_2(U)$
.1	.2481
.2	.3453
.3	.4162
.4	.4730
.5	.5207
1.0	.6825
1.5	.7795
2.0	.8426
2.5	.8857
3.0	.9170
3.5	.9385
4.0	.9540
4.5	.9664
5.0	.9745
5.5	.9805
6.0	.9860
6.5	.9900
7.0	.9914

TABLE 6.1.2

$F_2(U)$: $\nu = 2$, $\bar{\mu}^2 = 10$

U	$\bar{F}_2(U)$	$\bar{\beta}_1^2 = .25$		$\bar{\beta}_1^2 = 1$	
		$F_2(U)$	$F_2(U)-\bar{F}_2(U)$	$F_2(U)$	$F_2(U)-\bar{F}_2(U)$
.5	.2214	.2257	.0044	.2318	.0104
1.0	.3932	.4001	.0068	.4096	.0164
1.5	.5279	.5359	.0080	.5472	.0193
2.0	.6319	.6403	.0084	.6522	.0203
2.5	.7136	.7219	.0082	.7336	.0200
3.0	.7774	.7851	.0077	.7963	.0189
3.5	.8259	.8329	.0070	.8433	.0174
4.0	.8649	.8712	.0063	.8806	.0157
4.5	.8944	.9000	.0055	.9083	.0139
5.0	.9174	.9222	.0048	.9296	.0122
5.5	.9364	.9406	.0042	.9470	.0106
6.0	.9500	.9536	.0036	.9591	.0091
6.5	.9606	.9637	.0030	.9685	.0078
7.0	.9702	.9727	.0025	.9768	.0066
7.5	.9762	.9784	.0021	.9818	.0056
8.0	.9810	.9828	.0018	.9857	.0047
8.5	.9862	.9877	.0015	.9901	.0040
9.0	.9886	.9898	.0012	.9919	.0033
9.5	.9906	.9916	.0010	.9934	.0028

TABLE 6.1.3

$F_2(U)$: $\nu = 3$, $\bar{\mu}^2 = 10$

U	$\bar{F}_2(U)$	$\bar{\beta}_1^2 = .25$		$\bar{\beta}_1^2 = 1$	
		$F_2(U)$	$F_2(U)-\bar{F}_2(U)$	$F_2(U)$	$F_2(U)-\bar{F}_2(U)$
.5	.0811	.0855	.0044	.0921	.0110
1.0	.1986	.2083	.0097	.2228	.0242
1.5	.3179	.3318	.0138	.3526	.0346
2.0	.4274	.4440	.0166	.4689	.0415
2.5	.5249	.5429	.0181	.5700	.0452
3.0	.6083	.6268	.0185	.6545	.0463
3.5	.6787	.6969	.0182	.7241	.0454
4.0	.7389	.7562	.0173	.7821	.0432
4.5	.7875	.8035	.0161	.8276	.0401
5.0	.8284	.8430	.0146	.8650	.0366
5.5	.8619	.8751	.0132	.8948	.0329
6.0	.8880	.8997	.0117	.9172	.0292
6.5	.9106	.9209	.0103	.9362	.0256
7.0	.9279	.9368	.0089	.9502	.0223
7.5	.9419	.9496	.0077	.9612	.0193
8.0	.9544	.9610	.0066	.9709	.0165
8.5	.9630	.9687	.0056	.9771	.0141
9.0	.9700	.9748	.0048	.9820	.0120
9.5	.9771	.9812	.0040	.9872	.0101
10.0	.9811	.9845	.0034	.9896	.0085
10.5	.9854	.9883	.0028	.9926	.0071
11.0	.9888	.9912	.0024	.9948	.0059
11.5	.9903	.9923	.0020	.9953	.0050

TABLE 6.1.4

$F_2(U)$: $\nu = 4$, $\bar{\mu}^2 = 10$

U	$\bar{F}_2(U)$	$\bar{\beta}_1^2 = .25$		$\bar{\beta}_1^2 = 1$	
		$F_2(U)$	$F_2(U)-\bar{F}_2(U)$	$F_2(U)$	$F_2(U)-\bar{F}_2(U)$
.5	.0265	.0292	.0027	.0336	.0071
1.0	.0902	.0986	.0084	.1121	.0219
1.5	.1735	.1881	.0146	.2116	.0381
2.0	.2641	.2842	.0201	.3164	.0523
2.5	.3555	.3799	.0244	.4187	.0632
3.0	.4421	.4693	.0273	.5124	.0703
3.5	.5222	.5511	.0288	.5962	.0740
4.0	.5944	.6236	.0292	.6691	.0747
4.5	.6571	.6858	.0287	.7301	.0730
5.0	.7129	.7404	.0275	.7825	.0696
5.5	.7601	.7859	.0258	.8251	.0650
6.0	.8003	.8241	.0238	.8600	.0598
6.5	.8356	.8573	.0217	.8897	.0541
7.0	.8638	.8833	.0195	.9122	.0484
7.5	.8885	.9058	.0173	.9314	.0429
8.0	.9090	.9243	.0153	.9466	.0376
8.5	.9247	.9381	.0134	.9574	.0327
9.0	.9392	.9509	.0116	.9674	.0282
9.5	.9500	.9601	.0101	.9742	.0242
10.0	.9590	.9676	.0086	.9795	.0205
10.5	.9676	.9750	.0074	.9851	.0175
11.0	.9731	.9794	.0063	.9879	.0148
11.5	.9778	.9831	.0053	.9902	.0124
12.0	.9832	.9877	.0045	.9936	.0104
12.5	.9856	.9894	.0038	.9943	.0087
13.0	.9890	.9922	.0032	.9962	.0072
13.5	.9916	.9942	.0027	.9975	.0060

TABLE 6.1.5

$F_2(U)$: $\nu = 5$, $\bar{\mu}^2 = 10$

U	$\bar{F}_2(U)$	$\bar{\beta}_1^2 = .25$		$\bar{\beta}_1^2 = 1$	
		$F_2(U)$	$F_2(U)-\bar{F}_2(U)$	$F_2(U)$	$F_2(U)-\bar{F}_2(U)$
.5	.0079	.0091	.0013	.0114	.0035
1.0	.0374	.0430	.0055	.0528	.0153
1.5	.0869	.0988	.0118	.1194	.0325
2.0	.1507	.1696	.0188	.2019	.0512
2.5	.2236	.2491	.0255	.2923	.0687
3.0	.2999	.3310	.0311	.3832	.0833
3.5	.3767	.4121	.0354	.4707	.0940
4.0	.4505	.4887	.0382	.5513	.1008
4.5	.5198	.5595	.0397	.6236	.1039
5.0	.5844	.6244	.0400	.6881	.1037
5.5	.6418	.6812	.0393	.7429	.1010
6.0	.6939	.7318	.0378	.7903	.0963
6.5	.7400	.7758	.0358	.8302	.0903
7.0	.7790	.8123	.0333	.8622	.0833
7.5	.8143	.8449	.0306	.8901	.0759
8.0	.8435	.8714	.0279	.9119	.0683
8.5	.8686	.8937	.0251	.9295	.0609
9.0	.8914	.9138	.0224	.9453	.0538
9.5	.9089	.9288	.0198	.9562	.0472
10.0	.9250	.9425	.0175	.9661	.0411
10.5	.9385	.9537	.0153	.9740	.0355
11.0	.9481	.9614	.0133	.9787	.0305
11.5	.9580	.9695	.0115	.9841	.0261
12.0	.9650	.9748	.0099	.9872	.0222
12.5	.9708	.9792	.0085	.9896	.0188
13.0	.9771	.9843	.0072	.9929	.0158
13.5	.9805	.9867	.0061	.9938	.0133
14.0	.9846	.9898	.0052	.9957	.0111
14.5	.9879	.9923	.0044	.9972	.0093
15.0	.9892	.9929	.0037	.9969	.0077

TABLE 6.1.6

$F_2(U)$: $\nu = 1$, $\bar{\beta}_1^{\,2} = .25$, $\bar{\mu}^2 = 10$

U	$\bar{F}_2(U)$	$F_2(U)$	$F_2(U)-\bar{F}_2(U)$
.1	.2481	.2481	-.0000
.2	.3453	.3452	-.0001
.3	.4162	.4160	-.0002
.4	.4730	.4728	-.0002
.5	.5207	.5204	-.0003
1.0	.6825	.6825	.0001
2.0	.8426	.8417	-.0009
3.0	.9170	.9158	-.0012
4.0	.9540	.9540	.0000
5.0	.9745	.9743	-.0002
6.0	.9860	.9862	.0002
7.0	.9914	.9925	.0012

TABLE 6.1.7

$F_2(\nu)-\bar{F}_2(\nu)$

$\bar{\mu}^2$	$\bar{\beta}_1^2 = .25$	$\bar{\beta}_1^2 = 1$	$\bar{\beta}_1^2 = .25$	$\bar{\beta}_1^2 = 1$	$\bar{\beta}_1^2 = .25$	$\bar{\beta}_1^2 = 1$	$\bar{\beta}_1^2 = .25$	$\bar{\beta}_1^2 = 1$
	$F_2(2)-\bar{F}_2(2)$	$F_2(2)-\bar{F}_2(2)$	$F_2(3)-\bar{F}_2(3)$	$F_2(3)-\bar{F}_2(3)$	$F_2(4)-\bar{F}_2(4)$	$F_2(4)-\bar{F}_2(4)$	$F_2(5)-\bar{F}_2(5)$	$F_2(5)-\bar{F}_2(5)$
10	.0084	.0203	.0185	.0463	.0292	.0747	.0400	.1037
20	.0039	.0096	.0093	.0231	.0155	.0390	.0222	.0564
30	.0025	.0063	.0062	.0154	.0105	.0264	.0153	.0386
40	.0019	.0047	.0046	.0116	.0079	.0199	.0117	.0294
50	.0015	.0037	.0037	.0093	.0064	.0160	.0094	.0237
60	.0012	.0031	.0031	.0077	.0053	.0134	.0079	.0198
70	.0011	.0027	.0026	.0066	.0046	.0115	.0068	.0171
80	.0009	.0023	.0023	.0058	.0040	.0101	.0060	.0150
90	.0008	.0021	.0021	.0051	.0036	.0089	.0053	.0133
100	.0007	.0019	.0019	.0046	.0032	.0081	.0048	.0120
150	.0005	.0012	.0012	.0031	.0022	.0054	.0032	.0081
200	.0004	.0009	.0009	.0023	.0016	.0040	.0024	.0061
250	.0003	.0007	.0007	.0019	.0013	.0032	.0019	.0049
300	.0002	.0006	.0006	.0015	.0011	.0027	.0016	.0040
350	.0002	.0005	.0005	.0013	.0009	.0023	.0014	.0035
400	.0002	.0005	.0005	.0012	.0008	.0020	.0012	.0030
450	.0002	.0004	.0004	.0010	.0007	.0018	.0011	.0027
500	.0001	.0004	.0004	.0009	.0006	.0016	.0010	.0024

TABLE 6.1.8

Methods of Moments: $\bar{\mu}^2 = 10$

ν	$\beta_1^{\,2}$	$E(V_2)$	$E(V_2^{\,2})$	$\bar{a}$	$\bar{b}$	$\bar{\bar{a}}$	$\bar{\bar{b}}$
1	.25	1.248	4.667	.5003	1.995	.5000	1.996
2	.25	2.441	11.90	1.001	1.950	1.000	1.953
3	.25	3.599	21.57	1.503	1.916	1.500	1.919
4	.25	4.732	33.56	2.004	1.889	2.000	1.893
5	.25	5.846	47.82	2.504	1.867	2.500	1.871
1	1.00	1.991	11.83	.5038	1.976	.5000	1.991
2	1.00	3.763	28.10	1.016	1.852	1.000	1.882
3	1.00	5.396	48.13	1.532	1.762	1.500	1.799
4	1.00	6.935	71.43	2.061	1.682	2.000	1.734
5	1.00	8.393	97.69	2.585	1.624	2.500	1.679

TABLE 6.1.9a

G(U): $(\bar{a},\bar{b})$, $\nu = 2$, $\bar{\mu}^2 = 10$

U	$\bar{\beta}_1^{\,2} = .25$		$\bar{\beta}_1^{\,2} = 1$	
	G(U)	F_2(U)-G(U)	G(U)	F_2(U)-G(U)
.5	.2258	-.0000	.2301	.0016
1.0	.4003	-.0002	.4091	.0005
2.0	.6407	-.0003	.6535	-.0012
3.0	.7856	-.0004	.7972	-.0009
4.0	.8715	-.0003	.8820	-.0014
5.0	.9222	.0001	.9310	-.0014
6.0	.9535	.0001	.9590	.0002
7.0	.9729	-.0001	.9761	.0007
8.0	.9837	-.0009	.9868	-.0010
9.0	.9896	.0002	.9922	-.0003

TABLE 6.1.9b

G(U): $(\bar{\bar{a}},\bar{\bar{b}})$, $\nu = 2$, $\bar{\mu}^2 = 10$

U	$\bar{\beta}_1^{\,2} = .25$		$\bar{\beta}_1^{\,2} = 1$	
	G(U)	F_2(U)-G(U)	G(U)	F_2(U)-G(U)
.5	.2261	-.0004	.2336	-.0018
1.0	.4005	-.0005	.4119	-.0023
2.0	.6408	-.0004	.6543	-.0021
3.0	.7855	-.0004	.7968	-.0005
4.0	.8714	-.0002	.8811	-.0005
5.0	.9221	.0001	.9301	-.0005
6.0	.9534	.0001	.9583	.0008
7.0	.9728	-.0001	.9755	.0012
8.0	.9836	-.0008	.9862	-.0004
9.0	.9896	.0002	.9918	.0001

TABLE 6.1.10a

G(U): $(\bar{a},\bar{b})$, $\nu = 3$, $\bar{\mu}^2 = 10$

U	$\bar{\beta}_1^{\,2} = .25$		$\bar{\beta}_1^{\,2} = 1$	
	G(U)	F_2(U)-G(U)	G(U)	F_2(U)-G(U)
.5	.0855	.0000	.0903	.0018
1.0	.2086	-.0003	.2217	.0011
2.0	.4443	-.0003	.4702	-.0013
3.0	.6271	-.0004	.6564	-.0019
4.0	.7567	-.0006	.7835	-.0014
5.0	.8434	-.0003	.8660	-.0010
6.0	.8995	.0002	.9188	-.0016
7.0	.9367	.0002	.9509	-.0006
8.0	.9613	-.0003	.9705	.0004
9.0	.9758	-.0010	.9817	.0003
10.0	.9840	.0005	.9891	.0005
11.0	.9902	.0010	.9944	.0004

TABLE 6.1.10b

G(U): $(\bar{\bar{a}},\bar{\bar{b}})$, $\nu = 3$, $\bar{\mu}^2 = 10$

U	$\bar{\beta}_1^{\,2} = .25$		$\bar{\beta}_1^{\,2} = 1$	
	G(U)	F_2(U)-G(U)	G(U)	F_2(U)-G(U)
.5	.0857	-.0002	.0936	-.0015
1.0	.2089	-.0007	.2258	-.0031
2.0	.4446	-.0006	.4723	-.0034
3.0	.6272	-.0005	.6570	-.0025
4.0	.7567	-.0005	.7828	-.0007
5.0	.8433	-.0002	.8655	-.0006
6.0	.8993	.0003	.9173	-.0001
7.0	.9366	.0003	.9495	.0007
8.0	.9612	-.0002	.9687	.0021
9.0	.9757	-.0009	.9812	.0008
10.0	.9840	.0005	.9896	.0001
11.0	.9902	.0010	.9938	.0010

TABLE 6.1.11a

G(U): $(\bar{a},\bar{b})$, $\nu = 4$, $\bar{\mu}^2 = 10$

U	$\bar{\beta}_1^{\,2} = .25$ G(U)	$\bar{\beta}_1^{\,2} = .25$ $F_2(U)$-G(U)	$\bar{\beta}_1^{\,2} = 1$ G(U)	$\bar{\beta}_1^{\,2} = 1$ $F_2(U)$-G(U)
.5	.0292	.0000	.0318	.0018
1.0	.0986	-.0001	.1094	.0027
2.0	.2845	-.0003	.3154	.0010
3.0	.4698	-.0005	.5136	-.0012
4.0	.6236	.0000	.6707	-.0016
5.0	.7411	-.0001	.7837	-.0012
6.0	.8253	-.0012	.8616	-.0015
7.0	.8840	-.0007	.9130	-.0008
8.0	.9234	.0009	.9461	.0006
9.0	.9504	.0005	.9678	-.0003
10.0	.9687	-.0011	.9805	-.0010
11.0	.9800	-.0005	.9883	-.0004
12.0	.9866	.0011	.9930	.0006
13.0	.9916	.0006	.9952	.0010

TABLE 6.1.11b

G(U): $(\bar{\bar{a}},\bar{\bar{b}})$, $\nu = 4$, $\bar{\mu}^2 = 10$

U	$\bar{\beta}_1^{\,2} = .25$ G(U)	$\bar{\beta}_1^{\,2} = .25$ $F_2(U)$-G(U)	$\bar{\beta}_1^{\,2} = 1$ G(U)	$\bar{\beta}_1^{\,2} = 1$ $F_2(U)$-G(U)
.5	.0293	-.0001	.0344	-.0008
1.0	.0989	-.0004	.1143	-.0022
2.0	.2849	-.0006	.3206	-.0042
3.0	.4700	-.0007	.5162	-.0038
4.0	.6237	-.0000	.6703	-.0012
5.0	.7411	-.0006	.7825	.0001
6.0	.8251	-.0011	.8596	.0004
7.0	.8829	.0003	.9119	.0004
8.0	.9233	.0011	.9449	.0018
9.0	.9502	.0006	.9659	.0016
10.0	.9686	-.0010	.9791	.0005
11.0	.9799	-.0004	.9866	.0013
12.0	.9865	.0011	.9918	.0018
13.0	.9915	.0006	.9951	.0011

TABLE 6.1.12a

$G(U)$: $(\bar{a},\bar{b})$, $\nu = 5$, $\bar{\mu}^2 = 10$

U	$\bar{\beta}_1^2 = .25$		$\bar{\beta}_1^2 = 1$	
	G(U)	$F_2(U)$-G(U)	G(U)	$F_2(U)$-G(U)
.5	.0091	.0000	.0105	.0009
1.0	.0430	-.0001	.0506	.0021
2.0	.1700	-.0005	.2001	.0018
3.0	.3315	-.0005	.3834	-.0003
4.0	.4894	-.0006	.5532	-.0019
5.0	.6247	-.0003	.6903	-.0022
6.0	.7326	-.0008	.7924	-.0021
7.0	.8134	-.0011	.8643	-.0020
8.0	.8721	-.0007	.9131	-.0012
9.0	.9130	.0008	.9445	.0008
10.0	.9420	.0005	.9653	.0008
11.0	.9625	-.0011	.9786	.0001
12.0	.9754	-.0006	.9869	.0003
13.0	.9840	.0003	.9920	.0009
14.0	.9891	.0007	.9961	-.0003
15.0	.9930	-.0002	.9978	-.0009

TABLE 6.1.12b

$G(U)$: $(\bar{\bar{a}},\bar{\bar{b}})$, $\nu = 5$, $\bar{\mu}^2 = 10$

U	$\bar{\beta}_1^2 = .25$		$\bar{\beta}_1^2 = 1$	
	G(U)	$F_2(U)$-G(U)	G(U)	$F_2(U)$-G(U)
.5	.0092	-.0001	.0018	-.0004
1.0	.0432	-.0002	.0543	-.0016
2.0	.1703	-.0008	.2061	-.0041
3.0	.3318	-.0008	.3880	-.0048
4.0	.4895	-.0008	.5550	-.0038
5.0	.6247	-.0003	.6898	-.0016
6.0	.7326	-.0008	.7896	.0006
7.0	.8132	-.0010	.8611	.0012
8.0	.8719	-.0005	.9100	.0019
9.0	.9129	.0009	.9426	.0027
10.0	.9419	.0006	.9638	.0023
11.0	.9624	-.0010	.9782	.0005
12.0	.9753	-.0005	.9866	.0005
13.0	.9830	.0013	.9918	.0011
14.0	.9891	.0007	.9951	.0007
15.0	.9930	-.0001	.9963	.0005

6.2 Tabulations of the Exact Finite Sample Distribution Function Associated With $\underline{V}_3$

In this section we present tabulations of $\bar{F}_3(U)$, $F_3(U)$, and $F_3(U)-\bar{F}_3(U)$ for $\nu = 0, 1, 2, 3, 4, 5$, $\bar{\beta}_1^{\,2} = 0, .25, 1$, and $\bar{\mu}^2 = 10$ where the tabulations are carried out according to (4.2.2) and (5.2.10) (Tables 6.2.1 - 6.2.6). We tabulate $F_3(U)$ and $F_3(U)-\bar{F}_3(U)$ at intervals of U of $\chi^2_{.99}(m)/10$ for $\nu = 0, 1, 2, 3, 4, 5$, $\bar{\beta}_1^{\,2} = 0, .25, .81$, $m = 10$, and $\bar{\mu}^2 = 10$ according to (4.2.1) and (4.2.2) (Tables 6.2.7 - 6.2.8) for purpose of comparison with the results obtained by the aforementioned method. For all values of ν, $\bar{\beta}_1^{\,2}$, m, $\bar{\mu}^2$, and U which were investigated the computation employing (5.2.10) was significantly faster than that making use of (4.2.1) although the use of (5.2.10) excludes terms of order greater than one in $\bar{\mu}^{-2}$ so the latter computational method may be preferred for reason of accuracy for smaller $\bar{\mu}^2$'s. In particular, notice the computational problem in Table 6.2.6 for $\bar{\beta}_1^{\,2} = 1$ when $\bar{\mu}^2 = 10$.

From (5.2.11) we obtained that the extreme values of $F_3(U)-\bar{F}_3(U)$ are the solution of a quadratic (linear if $\bar{\beta}_1^{\,2} = 0$) equation. In Table 6.2.9 we present tabulations of $F_3(U^*)-\bar{F}_3(U^*)$ where U^* is a member of the set of the values of U solving the aforementioned quadratic (linear if $\bar{\beta}_1^{\,2} = 0$) equation. The tabulations are made for $\nu = 0, 1, 2, 3, 4, 5$, $\bar{\beta}_1^{\,2} = 0, .25, 1$, and $\bar{\mu}^2 = 10$. Values of $F_3(U^*)-\bar{F}_3(U^*)$ for other values of $\bar{\mu}^2$ are obtained by multiplying the computed values by $10/\bar{\mu}^2$.

If, in a given application, we have specified that we use $\bar{F}_3(U)$ to approximate $F_3(U)$ only if $| F_3(U)-\bar{F}_3(U) | < \epsilon$, $\epsilon > 0$, and having estimated $\bar{\beta}_1^{\,2}$ by $\hat{\bar{\beta}}_1^{\,2}$ and $\bar{\mu}^2$ by $\hat{\bar{\mu}}^2$ we obtain that $| F_3(U^*) - \bar{F}_3(U^*) | \geq \epsilon$, we may compute $F_3(U)$ for ν, m, $\hat{\bar{\beta}}_1^{\,2}$, and $\hat{\bar{\mu}}^2$. Whether we use a computed $F_3(U)$ or use $\bar{F}_3(U)$ as an approximation, the selected distribution function is then employed in tests of hypotheses involving ω_{11}.

TABLE 6.2.1

$F_3(U)$: $\nu = 0$, $m = 10$, $\bar{\mu}^2 = 10$

U	$\bar{F}_3(U)$	$\bar{\beta}_1^2 = 0$		$\bar{\beta}_1^2 = .25$		$\bar{\beta}_1^2 = 1$	
		$F_3(U)$	$F_3(U)-\bar{F}_3(U)$	$F_3(U)$	$F_3(U)-\bar{F}_3(U)$	$F_3(U)$	$F_3(U)-\bar{F}_3(U)$
1.0000	.0002	.0001	-.0001	.0002	.0001	.0004	.0003
2.0000	.0037	.0021	-.0015	.0046	.0009	.0083	.0046
2.3209	.0068	.0040	-.0027	.0082	.0015	.0146	.0078
3.0000	.0186	.0115	-.0071	.0214	.0028	.0362	.0176
4.0000	.0526	.0346	-.0180	.0562	.0036	.0887	.0361
4.6418	.0862	.0586	-.0275	.0882	.0020	.1324	.0463
5.0000	.1088	.0754	-.0334	.1088	.0000	.1589	.0501
6.0000	.1847	.1343	-.0504	.1746	-.0102	.2351	.0504
6.9627	.2708	.2052	-.0656	.2450	-.0257	.3048	.0340
7.0000	.2743	.2082	-.0661	.2478	-.0264	.3073	.0330
8.0000	.3709	.2927	-.0781	.3240	-.0469	.3709	.0000
9.0000	.4676	.3822	-.0854	.3993	-.0683	.4249	-.0427
9.2836	.4942	.4077	-.0866	.4201	-.0742	.4387	-.0556
10.0000	.5588	.4711	-.0877	.4711	-.0877	.4711	-.0877
11.0000	.6419	.5562	-.0857	.5390	-.1028	.5133	-.1286
11.6045	.6866	.6038	-.0828	.5773	-.1093	.5374	-.1492
12.0000	.7144	.6341	-.0803	.6020	-.1124	.5538	-.1606
13.0000	.7752	.7025	-.0727	.6589	-.1163	.5935	-.1817
13.9254	.8227	.7581	-.0645	.7075	-.1152	.6315	-.1912
14.0000	.8261	.7622	-.0639	.7111	-.1149	.6345	-.1916
15.0000	.8671	.8125	-.0547	.7578	-.1094	.6757	-.1914
16.0000	.8997	.8539	-.0458	.7990	-.1008	.7165	-.1832
16.2463	.9064	.8627	-.0437	.8081	-.0983	.7262	-.1802
17.0000	.9243	.8867	-.0376	.8341	-.0903	.7551	-.1693
18.0000	.9440	.9136	-.0304	.8650	-.0789	.7922	-.1518

TABLE 6.2.1 (cont'd.)

U	$\bar{F}_3(U)$	$\bar{\beta}_1^2 = 0$		$\bar{\beta}_1^2 = .25$		$\bar{\beta}_1^2 = 1$	
		$F_3(U)$	$F_3(U)-\bar{F}_3(U)$	$F_3(U)$	$F_3(U)-\bar{F}_3(U)$	$F_3(U)$	$F_3(U)-\bar{F}_3(U)$
18.5672	.9524	.9257	-.0267	.8799	-.0725	.8113	-.1411
19.0000	.9589	.9347	-.0241	.8913	-.0676	.8261	-.1327
20.0000	.9692	.9502	-.0189	.9124	-.0567	.8557	-.1135
20.8881	.9769	.9618	-.0151	.9290	-.0479	.8797	-.0972
21.0000	.9776	.9630	-.0146	.9308	-.0469	.8824	-.0952
22.0000	.9839	.9726	-.0112	.9457	-.0381	.9054	-.0785
23.0000	.9874	.9789	-.0085	.9569	-.0306	.9238	-.0637
23.2090	.9890	.9810	-.0080	.9599	-.0291	.9282	-.0608
24.0000	.9909	.9846	-.0064	.9667	-.0242	.9400	-.0510
25.0000	.9935	.9887	-.0047	.9745	-.0190	.9532	-.0403

TABLE 6.2.2

$F_3(U)$: $\nu = 1$, $m = 10$, $\bar{\mu}^2 = 10$

U	$\bar{F}_3(U)$	$\bar{\beta}_1^2 = 0$		$\bar{\beta}_1^2 = .25$		$\bar{\beta}_1^2 = 1$	
		$F_3(U)$	$F_3(U)-\bar{F}_3(U)$	$F_3(U)$	$F_3(U)-\bar{F}_3(U)$	$F_3(U)$	$F_3(U)-\bar{F}_3(U)$
1.0000	.0002	.0001	-.0001	.0003	.0001	.0005	.0004
2.0000	.0037	.0021	-.0015	.0052	.0015	.0098	.0061
2.3209	.0068	.0040	-.0027	.0093	.0026	.0173	.0105
3.0000	.0186	.0115	-.0071	.0242	.0056	.0433	.0247
4.0000	.0526	.0346	-.0180	.0634	.0108	.1067	.0541
4.6418	.0862	.0586	-.0275	.0992	.0130	.1600	.0738
5.0000	.1088	.0754	-.0334	.1221	.0134	.1923	.0835
6.0000	.1847	.1343	-.0504	.1947	.0101	.2855	.1008
6.6927	.2708	.2052	-.0656	.2713	.0005	.3703	.0996
7.0000	.2743	.2082	-.0661	.2743	.0000	.3734	.0991
8.0000	.3709	.2927	-.0781	.3553	-.0156	.4490	.0781
9.0000	.4676	.3822	-.0854	.4335	-.0342	.5104	.0427
9.2836	.4942	.4077	-.0866	.4547	-.0395	.5252	.0310
10.0000	.5588	.4711	-.0877	.5062	-.0526	.5588	.0000
11.0000	.6419	.5562	-.0857	.5733	-.0686	.5990	-.0429
11.6045	.6866	.6038	-.0828	.6104	-.0762	.6202	-.0664
12.0000	.7144	.6341	-.0803	.6341	-.0803	.6341	-.0803
13.0000	.7752	.7025	-.0727	.6879	-.0872	.6661	-.1090
13.9254	.8227	.7581	-.0645	.7333	-.0894	.6960	-.1267
14.0000	.8261	.7622	-.0639	.7366	-.0894	.6983	-.1277
15.0000	.8671	.8125	-.0547	.7796	-.0875	.7304	-.1367
16.0000	.8997	.8539	-.0458	.8173	-.0824	.7623	-.1374
16.2463	.9064	.8627	-.0437	.8256	-.0808	.7699	-.1365
17.0000	.9243	.8867	-.0376	.8491	-.0752	.7927	-.1317
18.0000	.9440	.9136	-.0304	.8772	-.0668	.8225	-.1215
18.5672	.9524	.9257	-.0267	.8906	-.0618	.8380	-.1144

TABLE 6.2.2 (cont'd.)

U	$\bar{F}_3(U)$	$\bar{\beta}_1^2 = 0$		$\bar{\beta}_1^2 = .25$		$\bar{\beta}_1^2 = 1$	
		$F_3(U)$	$F_3(U)-\bar{F}_3(U)$	$F_3(U)$	$F_3(U)-\bar{F}_3(U)$	$F_3(U)$	$F_3(U)-\bar{F}_3(U)$
19.0000	.9589	.9347	-.0241	.9010	-.0579	.8503	-.1086
20.0000	.9692	.9502	-.0189	.9200	-.0492	.8746	-.0946
20.8881	.9769	.9618	-.0151	.9350	-.0419	.8948	-.0821
21.0000	.9776	.9630	-.0146	.9366	-.0410	.8971	-.0805
22.0000	.9839	.9726	-.0112	.9502	-.0336	.9166	-.0672
23.0000	.9874	.9789	-.0085	.9603	-.0272	.9322	-.0552
23.2090	.9890	.9810	-.0080	.9631	-.0259	.9362	-.0528
24.0000	.9909	.9846	-.0064	.9693	-.0217	.9463	-.0446
25.0000	.9935	.9887	-.0047	.9764	-.0171	.9579	-.0355

TABLE 6.2.3

$F_3(U)$: $\nu = 2$, $m = 10$, $\bar{\mu}^2 = 10$

U	$\bar{F}_3(U)$	$\bar{\beta}_1^2 = 0$		$\bar{\beta}_1^2 = .25$		$\bar{\beta}_1^2 = 1$	
		$F_3(U)$	$F_3(U)-\bar{F}_3(U)$	$F_3(U)$	$F_3(U)-\bar{F}_3(U)$	$F_3(U)$	$F_3(U)-\bar{F}_3(U)$
1.0000	.0002	.0001	-.0001	.0003	.0001	.0006	.0004
2.0000	.0037	.0021	-.0015	.0058	.0021	.0113	.0077
2.3209	.0068	.0040	-.0027	.0104	.0037	.0201	.0133
3.0000	.0186	.0115	-.0071	.0271	.0085	.0503	.0318
4.0000	.0526	.0346	-.0180	.0707	.0180	.1248	.0722
4.6418	.0862	.0586	-.0275	.1102	.0240	.1875	.1014
5.0000	.1088	.0754	-.0334	.1355	.0267	.2257	.1169
6.0000	.1847	.1343	-.0504	.2149	.0302	.3359	.1512
6.9627	.2708	.2052	-.0656	.2975	.0267	.4359	.1651
7.0000	.2743	.2082	-.0661	.3007	.0264	.4395	.1652
8.0000	.3709	.2927	-.0781	.3865	.0156	.5272	.1563
9.0000	.4676	.3822	-.0854	.4676	.0000	.5958	.1281
9.2836	.4942	.4077	-.0866	.4893	-.0049	.6118	.1176
10.0000	.5588	.4711	-.0877	.5413	-.0175	.6465	.0877
11.0000	.6419	.5562	-.0857	.6076	-.0343	.6847	.0429
11.6045	.6866	.6038	-.0828	.6435	-.0431	.7030	.0164
12.0000	.7144	.6341	-.0803	.6662	-.0482	.7144	.0000
13.0000	.7752	.7025	-.0727	.7170	-.0581	.7388	-.0363
13.9254	.8227	.7581	-.0645	.7591	-.0636	.7605	-.0621
14.0000	.8261	.7622	-.0639	.7622	-.0639	.7622	-.0639
15.0000	.8671	.8125	-.0547	.8015	-.0656	.7851	-.0820
16.0000	.8997	.8539	-.0458	.8356	-.0641	.8081	-.0916
16.2463	.9064	.8627	-.0437	.8431	-.0633	.8136	-.0928
17.0000	.9243	.8867	-.0376	.8642	-.0602	.8303	-.0940
18.0000	.9440	.9136	-.0304	.8893	-.0547	.8529	-.0911

TABLE 6.2.3 (cont'd.)

U	$\bar{F}_3(U)$	$\bar{\beta}_1^2 = 0$		$\bar{\beta}_1^2 = .25$		$\bar{\beta}_1^2 = 1$	
		$F_3(U)$	$F_3(U)-\bar{F}_3(U)$	$F_3(U)$	$F_3(U)-\bar{F}_3(U)$	$F_3(U)$	$F_3(U)-\bar{F}_3(U)$
18.5672	.9524	.9257	-.0267	.9013	-.0511	.8647	-.0877
19.0000	.9589	.9347	-.0241	.9106	-.0483	.8744	-.0845
20.0000	.9692	.9502	-.0189	.9275	-.0416	.8935	-.0757
20.8881	.9769	.9618	-.0151	.9411	-.0358	.9099	-.0670
21.0000	.9776	.9630	-.0146	.9425	-.0351	.9117	-.0659
22.0000	.9839	.9726	-.0112	.9547	-.0291	.9278	-.0560
23.0000	.9874	.9789	-.0085	.9637	-.0238	.9407	-.0467
23.2090	.9890	.9810	-.0080	.9663	-.0227	.9442	-.0448
24.0000	.9909	.9846	-.0064	.9718	-.0191	.9527	-.0382
25.0000	.9935	.9887	-.0047	.9783	-.0152	.9627	-.0308

TABLE 6.2.4

$F_3(U)$: $\nu = 3$, $m = 10$, $\bar{\mu}^2 = 10$

U	$\bar{F}_3(U)$	$\bar{\beta}_1^2 = 0$		$\bar{\beta}_1^2 = .25$		$\bar{\beta}_1^2 = 1$	
		$F_3(U)$	$F_3(U)-\bar{F}_3(U)$	$F_3(U)$	$F_3(U)-\bar{F}_3(U)$	$F_3(U)$	$F_3(U)-\bar{F}_3(U)$
1.0000	.0002	.0001	-.0001	.0003	.0002	.0007	.0005
2.0000	.0037	.0021	-.0015	.0064	.0028	.0129	.0092
2.3209	.0068	.0040	-.0027	.0115	.0048	.0228	.0160
3.0000	.0186	.0115	-.0071	.0299	.0113	.0574	.0388
4.0000	.0526	.0346	-.0180	.0779	.0253	.1428	.0902
4.6418	.0862	.0586	-.0275	.1212	.0350	.2151	.1289
5.0000	.1088	.0754	-.0334	.1488	.0401	.2591	.1503
6.0000	.1847	.1343	-.0504	.2351	.0504	.3863	.2016
6.9627	.2708	.2052	-.0656	.3237	.0529	.5014	.2307
7.0000	.2743	.2082	-.0661	.3271	.0529	.5056	.2313
8.0000	.3709	.2927	-.0781	.4178	.0469	.6053	.2344
9.0000	.4676	.3822	-.0854	.5018	.0342	.6812	.2135
9.2836	.4942	.4077	-.0866	.5240	.0297	.6984	.2041
10.0000	.5588	.4711	-.0877	.5764	.0175	.7343	.1755
11.0000	.6419	.5562	-.0857	.6419	.0000	.7704	.1286
11.6045	.6866	.6038	-.0828	.6766	-.0100	.7857	.0991
12.0000	.7144	.6341	-.0803	.6984	-.0161	.7947	.0803
13.0000	.7752	.7025	-.0727	.7461	-.0291	.8115	.0363
13.9254	.8227	.7581	-.0645	.7849	-.0378	.8251	.0024
14.0000	.8261	.7622	-.0639	.7877	-.0383	.8261	.0000
15.0000	.8671	.8125	-.0547	.8234	-.0437	.8398	-.0273
16.0000	.8997	.8539	-.0458	.8539	-.0458	.8539	-.0458
16.2463	.9064	.8627	-.0437	.8606	-.0459	.8573	-.0491
17.0000	.9243	.8867	-.0376	.8792	-.0451	.8679	-.0564
18.0000	.9440	.9136	-.0304	.9015	-.0425	.8833	-.0607

TABLE 6.2.4 (cont'd.)

U	$\bar{F}_3(U)$	$\bar{\beta}_1^2 = 0$		$\bar{\beta}_1^2 = .25$		$\bar{\beta}_1^2 = 1$	
		$F_3(U)$	$F_3(U)-\bar{F}_3(U)$	$F_3(U)$	$F_3(U)-\bar{F}_3(U)$	$F_3(U)$	$F_3(U)-\bar{F}_3(U)$
18.5672	.9524	.9257	-.0267	.9120	-.0404	.8914	-.0610
19.0000	.9589	.9347	-.0241	.9203	-.0386	.8985	-.0603
20.0000	.9692	.9502	-.0189	.9351	-.0340	.9124	-.0567
20.8881	.9769	.9618	-.0151	.9471	-.0298	.9250	-.0519
21.0000	.9776	.9630	-.0146	.9483	-.0293	.9264	-.0513
22.0000	.9839	.9726	-.0112	.9592	-.0247	.9390	-.0448
23.0000	.9874	.9789	-.0085	.9671	-.0204	.9492	-.0382
23.2090	.9890	.9810	-.0080	.9695	-.0195	.9522	-.0369
24.0000	.9909	.9846	-.0064	.9744	-.0166	.9591	-.0319
25.0000	.9935	.9887	-.0047	.9802	-.0133	.9674	-.0261

TABLE 6.2.5

$F_3(U)$: $\nu = 4$, $m = 10$, $\bar{\mu}^2 = 10$

U	$\bar{F}_3(U)$	$\bar{\beta}_1^2 = 0$		$\bar{\beta}_1^2 = .25$		$\bar{\beta}_1^2 = 1$	
		$F_3(U)$	$F_3(U)-\bar{F}_3(U)$	$F_3(U)$	$F_3(U)-\bar{F}_3(U)$	$F_3(U)$	$F_3(U)-\bar{F}_3(U)$
1.0000	.0002	.0001	-.0001	.0004	.0002	.0008	.0006
2.0000	.0037	.0021	-.0015	.0070	.0034	.0144	.0107
2.3209	.0068	.0040	-.0027	.0126	.0059	.0256	.0188
3.0000	.0186	.0115	-.0071	.0327	.0141	.0645	.0459
4.0000	.0526	.0346	-.0180	.0851	.0325	.1609	.1083
4.6418	.0862	.0586	-.0275	.1322	.0461	.2426	.1565
5.0000	.1088	.0754	-.0334	.1622	.0534	.2925	.1837
6.0000	.1847	.1343	-.0504	.2552	.0706	.4367	.2520
6.9627	.2708	.2052	-.0656	.3499	.0792	.5670	.2962
7.0000	.2743	.2082	-.0661	.3536	.0793	.5716	.2974
8.0000	.3709	.2927	-.0781	.4490	.0781	.6835	.3126
9.0000	.4676	.3822	-.0854	.5360	.0683	.7666	.2989
9.2836	.4942	.4077	-.0866	.5586	.0643	.7849	.2907
10.0000	.5588	.4711	-.0877	.6114	.0526	.8220	.2632
11.0000	.6419	.5562	-.0857	.6762	.0343	.8561	.2143
11.6045	.6866	.6038	-.0828	.7097	.0231	.8685	.1819
12.0000	.7144	.6341	-.0803	.7305	.0161	.8751	.1606
13.0000	.7752	.7025	-.0727	.7752	.0000	.8842	.1090
13.9254	.8227	.7581	-.0645	.8107	-.0119	.8896	.0669
14.0000	.8261	.7622	-.0639	.8133	-.0128	.8899	.0639
15.0000	.8671	.8125	-.0547	.8453	-.0219	.8945	.0273
16.0000	.8997	.8539	-.0458	.8722	-.0275	.8997	.0000
16.2463	.9064	.8627	-.0437	.8780	-.0284	.9010	-.0054
17.0000	.9243	.8867	-.0376	.8942	-.0301	.9055	-.0188
18.0000	.9440	.9136	-.0304	.9136	-.0304	.9136	-.0304

TABLE 6.2.5 (cont'd.)

U	$\bar{F}_3(U)$	$\bar{\beta}_1^2 = 0$		$\bar{\beta}_1^2 = .25$		$\bar{\beta}_1^2 = 1$	
		$F_3(U)$	$F_3(U)-\bar{F}_3(U)$	$F_3(U)$	$F_3(U)-\bar{F}_3(U)$	$F_3(U)$	$F_3(U)-\bar{F}_3(U)$
18.5672	.9524	.9257	-.0267	.9226	-.0297	.9181	-.0343
19.0000	.9589	.9347	-.0241	.9299	-.0290	.9227	-.0362
20.0000	.9692	.9502	-.0189	.9427	-.0265	.9313	-.0378
20.8881	.9769	.9618	-.0151	.9531	-.0238	.9401	-.0369
21.0000	.9776	.9630	-.0146	.9542	-.0234	.9410	-.0366
22.0000	.9839	.9726	-.0112	.9637	-.0202	.9502	-.0336
23.0000	.9874	.9789	-.0085	.9705	-.0170	.9577	-.0297
23.2090	.9890	.9810	-.0080	.9727	-.0163	.9602	-.0288
24.0000	.9909	.9846	-.0064	.9769	-.0140	.9654	-.0255
25.0000	.9935	.9887	-.0047	.9821	-.0114	.9721	-.0213

TABLE 6.2.6

$F_3(U)$: $\nu = 5$, $m = 10$, $\bar{\mu}^2 = 10$

U	$\bar{F}_3(U)$	$\bar{\beta}_1^2 = 0$		$\bar{\beta}_1^2 = .25$		$\bar{\beta}_1^2 = 1$	
		$F_3(U)$	$F_3(U)-\bar{F}_3(U)$	$F_3(U)$	$F_3(U)-\bar{F}_3(U)$	$F_3(U)$	$F_3(U)-\bar{F}_3(U)$
1.0000	.0002	.0001	-.0001	.0004	.0002	.0008	.0007
2.0000	.0037	.0021	-.0015	.0076	.0040	.0159	.0123
2.3209	.0068	.0040	-.0027	.0137	.0070	.0283	.0215
3.0000	.0186	.0115	-.0071	.0355	.0169	.0715	.0529
4.0000	.0526	.0346	-.0180	.0923	.0397	.1789	.1263
4.6418	.0862	.0586	-.0275	.1433	.0571	.2702	.1840
5.0000	.1088	.0754	-.0334	.1756	.0668	.3259	.2171
6.0000	.1847	.1343	-.0504	.2754	.0907	.4871	.3025
6.9627	.2708	.2052	-.0656	.3761	.1054	.6325	.3618
7.0000	.2743	.2082	-.0661	.3800	.1057	.6377	.3635
8.0000	.3709	.2927	-.0781	.4803	.1094	.7616	.3907
9.0000	.4676	.3822	-.0854	.5701	.1025	.8520	.3844
9.2836	.4942	.4077	-.0866	.5932	.0990	.8715	.3772
10.0000	.5588	.4711	-.0877	.6465	.0877	.9097	.3509
11.0000	.6419	.5562	-.0857	.7104	.0686	.9418	.3000
11.6045	.6866	.6038	-.0828	.7428	.0562	.9513	.2647
12.0000	.7144	.6341	-.0803	.7626	.0482	.9554	.2409
13.0000	.7752	.7025	-.0727	.8042	.0291	.9569	.1817
13.9254	.8227	.7581	-.0645	.8365	.0139	.9542	.1315
14.0000	.8261	.7622	-.0639	.8388	.0128	.9538	.1277
15.0000	.8671	.8125	-.0547	.8671	.0000	.9492	.0820
16.0000	.8997	.8539	-.0458	.8906	-.0092	.9455	.0458
16.2463	.9064	.8627	-.0437	.8955	-.0109	.9448	.0383
17.0000	.9243	.8867	-.0376	.9093	-.0150	.9431	.0188
18.0000	.9440	.9136	-.0304	.9258	-.0182	.9440	.0000

TABLE 6.2.6 (cont'd.)

U	$\bar{F}_3(U)$	$\bar{\beta}_1^2 = 0$		$\bar{\beta}_1^2 = .25$		$\bar{\beta}_1^2 = 1$	
		$F_3(U)$	$F_3(U)-\bar{F}_3(U)$	$F_3(U)$	$F_3(U)-\bar{F}_3(U)$	$F_3(U)$	$F_3(U)-\bar{F}_3(U)$
18.5672	.9524	.9257	-.0267	.9333	-.0191	.9448	-.0076
19.0000	.9589	.9347	-.0241	.9396	-.0193	.9468	-.0121
20.0000	.9692	.9502	-.0189	.9502	-.0189	.9502	-.0189
20.8881	.9769	.9618	-.0151	.9592	-.0178	.9551	-.0218
21.0000	.9776	.9630	-.0146	.9600	-.0176	.9557	-.0220
22.0000	.9839	.9726	-.0112	.9682	-.0157	.9614	-.0224
23.0000	.9874	.9789	-.0085	.9738	-.0136	.9662	-.0212
23.2090	.9890	.9810	-.0080	.9759	-.0131	.9682	-.0208
24.0000	.9909	.9846	-.0064	.9795	-.0115	.9718	-.0191
25.0000	.9935	.9887	-.0047	.9840	-.0095	.9769	-.0166

TABLE 6.2.7a

$F_3(U)$: $\bar{\beta}_1^{\,2} = 0$, $m = 10$, $\bar{\mu}^2 = 10$; $\nu = 0, 1, 2$

U	$\bar{F}_3(U)$	$\nu = 0$		$\nu = 1$		$\nu = 2$	
		$F_3(U)$	$F_3(U)-\bar{F}_3(U)$	$F_3(U)$	$F_3(U)-\bar{F}_3(U)$	$F_3(U)$	$F_3(U)-\bar{F}_3(U)$
2.3209	.0068	.0049	-.0019	.0050	-.0018	.0051	-.0017
4.6418	.0862	.0654	-.0208	.0670	-.0192	.0684	-.0178
6.9627	.2708	.2150	-.0558	.2198	-.0510	.2238	-.0470
9.2836	.4942	.4101	-.0841	.4180	-.0762	.4246	-.0696
11.6045	.6866	.5931	-.0935	.6037	-.0829	.6110	-.0756
13.9254	.8227	.7345	-.0882	.7464	-.0763	.7543	-.0684
16.2463	.9064	.8326	-.0738	.8440	-.0624	.8523	-.0541
18.5672	.9524	.8962	-.0562	.9051	-.0473	.9129	-.0395
20.8881	.9769	.9343	-.0426	.9434	-.0335	.9481	-.0288
23.2090	.9890	.9563	-.0327	.9649	-.0241	.9692	-.0198

TABLE 6.2.7b

$F_3(U)$: $\bar{\beta}_1^{\,2} = .25$, $m = 10$, $\bar{\mu}^2 = 10$; $\nu = 0, 1, 2$

U	$\bar{F}_3(U)$	$\nu = 0$		$\nu = 1$		$\nu = 2$	
		$F_3(U)$	$F_3(U)-\bar{F}_3(U)$	$F_3(U)$	$F_3(U)-\bar{F}_3(U)$	$F_3(U)$	$F_3(U)-\bar{F}_3(U)$
2.3209	.0068	.0073	.0005	.0081	.0013	.0087	.0019
4.6418	.0862	.0857	-.0005	.0934	.0072	.1000	.0138
6.9627	.2708	.2532	-.0176	.2735	.0027	.2898	.0190
9.2836	.4942	.4448	-.0494	.4754	-.0188	.5001	.0059

TABLE 6.2.7b (cont'd.)

U	$\bar{F}_3(U)$	$\nu = 0$		$\nu = 1$		$\nu = 2$	
		$F_3(U)$	$F_3(U)-\bar{F}_3(U)$	$F_3(U)$	$F_3(U)-\bar{F}_3(U)$	$F_3(U)$	$F_3(U)-\bar{F}_3(U)$
11.6045	.6866	.6089	-.0777	.6438	-.0428	.6723	-.0143
13.9254	.8227	.7289	-.0938	.7647	-.0580	.7918	-.0309
16.2463	.9064	.8106	-.0958	.8440	-.0624	.8683	-.0381
18.5672	.9524	.8636	-.0888	.8946	-.0578	.9145	-.0379
20.8881	.9769	.8991	-.0778	.9257	-.0512	.9431	-.0338
23.2090	.9890	.9229	-.0661	.9462	-.0428	.9608	-.0282

TABLE 6.2.7c

$F_3(U)$: $\bar{\beta}_1^{\,2} = .81$, $m = 10$, $\bar{\mu}^2 = 10$; $\nu = 0, 1, 2$

U	$\bar{F}_3(U)$	$\nu = 0$		$\nu = 1$		$\nu = 2$	
		$F_3(U)$	$F_3(U)-\bar{F}_3(U)$	$F_3(U)$	$F_3(U)-\bar{F}_3(U)$	$F_3(U)$	$F_3(U)-\bar{F}_3(U)$
2.3209	.0068	.0119	.0051	.0145	.0077	.0175	.0107
4.6418	.0862	.1140	.0278	.1362	.0500	.1571	.0709
6.9627	.2708	.2927	.0219	.3392	.0684	.3809	.1101
9.2836	.4942	.4703	-.0239	.5308	.0366	.5825	.0883
11.6045	.6866	.6107	-.0759	.6750	-.0116	.7254	.0388
13.9245	.8227	.7119	-.1108	.7732	-.0495	.8186	-.0041
16.2463	.9064	.7835	-.1229	.8384	-.0680	.8775	-.0289
18.5672	.9524	.8332	-.1192	.8820	-.0704	.9149	-.0375
20.8881	.9769	.8673	-.1096	.9105	-.0664	.9378	-.0391

TABLE 6.2.8a

$F_3(U)$: $\bar{\beta}_1^{\,2} = 0$, $m = 10$, $\bar{\mu}^2 = 10$; $\nu = 3, 4, 5$

U	$\bar{F}_3(U)$	ν = 3		ν = 4		ν = 5	
		$F_3(U)$	$F_3(U)-\bar{F}_3(U)$	$F_3(U)$	$F_3(U)-\bar{F}_3(U)$	$F_3(U)$	$F_3(U)-\bar{F}_3(U)$
2.3209	.0068	.0052	-.0016	.0053	-.0015	.0054	-.0014
4.6418	.0862	.0695	-.0167	.0705	-.0157	.0714	-.0148
6.9627	.2708	.2271	-.0437	.2300	-.0408	.2325	-.0383
9.2836	.4942	.4300	-.0642	.4347	-.0595	.4386	-.0556
11.6045	.6866	.6182	-.0684	.6227	-.0639	.6269	-.0597
13.9254	.8227	.7617	-.0610	.7665	-.0562	.7720	-.0507
16.2463	.9064	.8573	-.0491	.8626	-.0438	.8673	-.0391
18.5672	.9524	.9174	-.0350	.9224	-.0300	.9246	-.0278
20.8881	.9769	.9537	-.0232	.9566	-.0203	.9585	-.0184
23.2090	.9890	.9734	-.0156	.9751	-.0139	.9782	-.0108

TABLE 6.2.8b

$F_3(U)$: $\bar{\beta}_1^{\,2} = .25$, $m = 10$, $\bar{\mu}^2 = 10$; $\nu = 3, 4, 5$

U	$\bar{F}_3(U)$	ν = 3		ν = 4		ν = 5	
		$F_3(U)$	$F_3(U)-\bar{F}_3(U)$	$F_3(U)$	$F_3(U)-\bar{F}_3(U)$	$F_3(U)$	$F_3(U)-\bar{F}_3(U)$
2.3209	.0068	.0093	.0025	.0098	.0030	.0102	.0034
4.6418	.0862	.1056	.0194	.1105	.0243	.1148	.0286
6.9627	.2708	.3037	.0329	.3156	.0448	.3258	.0550
9.2836	.4942	.5203	.0261	.5372	.0430	.5519	.0577

TABLE 6.2.8b (cont'd.)

U	$\bar{F}_3(U)$	$\nu = 3$		$\nu = 4$		$\nu = 5$	
		$F_3(U)$	$F_3(U)-\bar{F}_3(U)$	$F_3(U)$	$F_3(U)-\bar{F}_3(U)$	$F_3(U)$	$F_3(U)-\bar{F}_3(U)$
11.6045	.6866	.6944	.0078	.7119	.0253	.7268	.0402
13.9254	.8227	.8127	-.0100	.8282	.0055	.8417	.0190
16.2463	.9064	.8863	-.0201	.8999	-.0065	.9099	.0035
18.5672	.9524	.9295	-.0229	.9411	-.0113	.9484	-.0040
20.8881	.9769	.9553	-.0216	.9647	-.0122	.9707	-.0062
23.2090	.9890	.9706	-.0184	.9784	-.0106	.9827	-.0063

TABLE 6.2.8c

$F_3(U)$: $\bar{\beta}_1^2 = .81$, $m = 10$, $\bar{\mu}^2 = 10$; $\nu = 3, 4, 5$

U	$\bar{F}_3(U)$	$\nu = 3$		$\nu = 4$		$\nu = 5$	
		$F_3(U)$	$F_3(U)-\bar{F}_3(U)$	$F_3(U)$	$F_3(U)-\bar{F}_3(U)$	$F_3(U)$	$F_3(U)-\bar{F}_3(U)$
2.3209	.0068	.0201	.0133	.0225	.0157	.0248	.0180
4.6418	.0862	.1759	.0897	.1931	.1069	.2090	.1228
6.9627	.2708	.4169	.1461	.4487	.1779	.4767	.2059
9.2836	.4942	.6248	.1306	.6611	.1669	.6919	.1977
11.6045	.6866	.7662	.0796	.7996	.1130	.8256	.1390
13.9254	.8227	.8536	.0309	.8806	.0579	.9013	.0786
16.2463	.9064	.9061	-.0003	.9273	.0209	.9427	.0363
18.5672	.9524	.9381	-.0143	.9543	.0019	.9656	.0132
20.8881	.9769	.9577	-.0192	.9708	-.0061	.9779	.0010

TABLE 6.2.9

$F_3(U^*)-\bar{F}_3(U^*)$: $m = 10$, $\bar{\mu}^2 = 10$

$\bar{\beta}_1^2$	$\nu = 0$		$\nu = 1$		$\nu = 2$		$\nu = 3$	
	U*	$F_3(U^*)-\bar{F}_3(U^*)$	U*	$F_3(U^*)-\bar{F}_3(U^*)$	U*	$F_3(U^*)-\bar{F}_3(U^*)$	U*	$F_3(U^*)-\bar{F}_3(U^*)$
0	10.0000	-.0877	10.0000	-.0877	10.0000	-.0877	10.0000	-.0877
.25	3.7830	.0037	5.0000	.0134	6.0000	.0302	6.7830	.0531
.25	13.2170	-.1164	14.0000	-.0894	15.0000	-.0656	16.2170	-.0459
1	5.5279	.0524	6.4174	.1026	7.1010	.1653	7.6148	.2363
1	14.4721	-.1926	15.5826	-.1380	16.8990	-.0941	18.3852	-.0610

TABLE 6.2.9 (cont'd.)

$\bar{\beta}_1^2$	$\nu = 4$		$\nu = 5$	
	U*	$F_3(U^*)-\bar{F}_3(U^*)$	U*	$F_3(U^*)-\bar{F}_3(U^*)$
0	10.0000	-.0877	10.0000	-.0877
.25	7.3765	.0800	7.8211	.1096
.25	17.6235	-.0305	19.1789	-.0193
1	8.0000	.3126	8.2918	.3921
1	20.0000	-.0378	21.7082	-.0225

6.3 Tabulations of the Exact Finite Sample Distribution Function Associated with V_4

In this section we present tabulations of $\bar{F}_4(U)$, $F_4(U)$, and $F_4(U)-\bar{F}_4(U)$ for $\nu = 0, 1, 2, 3, 4, 5$, $\bar{\beta}_1^2 = 0, .25, 1$, and $\bar{\mu}^2 = 10$ where the tabulations are carried out according to (4.3.2) and (5.3.17) (Tables 6.3.1 - 6.3.6). The computation described above is considerably faster than a similar computation which has been evaluated for a few values of ν, m, $\bar{\beta}_1^2$, $\bar{\mu}^2$, and U and makes use of (4.3.1) and (4.3.2) although the use of (5.3.17) excludes terms of order greater than one in $\bar{\mu}^{-2}$ so the latter computational method may be preferred for reason of accuracy for smaller $\bar{\mu}^2$'s. In particular, notice the computational problem in Table 6.3.6 for $\bar{\beta}_1^2 = 1$ when $\bar{\mu}^2 = 10$.

From (5.2.18) we obtained that the extreme values of $F_4(U)-\bar{F}_4(U)$ are the solution of a quadratic (linear if $\bar{\beta}_1^2 = 0$) equation. In Table 6.3.7 we present tabulations of $F_4(U^*)-\bar{F}_4(U^*)$ where U^* is a member of the set of the values of U solving the aforementioned quadratic (linear if $\bar{\beta}_1^2 = 0$) equation. The tabulations are made for $\nu = 0, 1, 2, 3, 4, 5$, $\bar{\beta}_1^2 = 0, .25, 1$, and $\bar{\mu}^2 = 10$. Values of $F_4(U^*)-\bar{F}_4(U^*)$ for other values of $\bar{\mu}^2$ are obtained by multiplying the computed values by $10/\bar{\mu}^2$.

If, in a given application, we have specified that we use $\bar{F}_4(U)$ to approximate $F_4(U)$ only if $|F_4(U)-\bar{F}_4(U)| < \epsilon$, $\epsilon > 0$, and having estimated $\bar{\beta}_1^2$ by $\hat{\bar{\beta}}_1^2$ and $\bar{\mu}^2$ by $\hat{\bar{\mu}}^2$ we obtain that $|F_4(U^*) - \bar{F}_4(U^*)| \geq \epsilon$, we may compute $F_4(U)$ for ν, m, $\hat{\bar{\beta}}_1^2$, and $\hat{\bar{\mu}}^2$. Whether we use a computed $F_4(U)$ or use $\overline{F}_4(U)$ as an approximation, the selected distribution function is then employed in tests of hypotheses involving ω_{11}.

TABLE 6.3.1

$F_4(U)$: $\nu = 0$, $m = 10$, $\bar{\mu}^2 = 10$

U	$\bar{F}_4(U)$	$\bar{\beta}_1^2 = 0$		$\bar{\beta}_1^2 = .25$		$\bar{\beta}_1^2 = 1$	
		$F_4(U)$	$F_4(U)-\bar{F}_4(U)$	$F_4(U)$	$F_4(U)-\bar{F}_4(U)$	$F_4(U)$	$F_4(U)-\bar{F}_4(U)$
1.0	.0002	.0001	-.0001	.0002	.0001	.0004	.0003
2.0	.0037	.0021	-.0015	.0046	.0009	.0083	.0046
3.0	.0186	.0115	-.0071	.0214	.0028	.0362	.0176
4.0	.0526	.0346	-.0180	.0562	.0036	.0887	.0361
5.0	.1088	.0754	-.0334	.1088	.0000	.1589	.0501
6.0	.1847	.1343	-.0504	.1746	-.0101	.2351	.0504
7.0	.2743	.2082	-.0661	.2478	-.0264	.3073	.0330
8.0	.3709	.2927	-.0781	.3240	-.0469	.3709	.0000
9.0	.4676	.3822	-.0854	.3993	-.0683	.4249	-.0427
10.0	.5588	.4711	-.0877	.4711	-.0877	.4711	-.0877
11.0	.6419	.5562	-.0857	.5390	-.1028	.5133	-.1286
12.0	.7144	.6341	-.0803	.6020	-.1124	.5538	-.1606
13.0	.7752	.7025	-.0727	.6589	-.1163	.5935	-.1817
14.0	.8261	.7622	-.0639	.7111	-.1149	.6345	-.1916
15.0	.8671	.8125	-.0547	.7578	-.1094	.6757	-.1914
16.0	.8997	.8539	-.0458	.7990	-.1008	.7165	-.1832
17.0	.9243	.8867	-.0376	.8341	-.0903	.7551	-.1693
18.0	.9440	.9136	-.0304	.8650	-.0789	.7922	-.1518
19.0	.9589	.9347	-.0241	.8913	-.0676	.8261	-.1327
20.0	.9692	.9502	-.0189	.9124	-.0567	.8557	-.1135
21.0	.9776	.9630	-.0146	.9308	-.0469	.8824	-.0952
22.0	.9839	.9726	-.0112	.9457	-.0381	.9054	-.0785
23.0	.9874	.9789	-.0085	.9569	-.0306	.9238	-.0637
24.0	.9909	.9846	-.0064	.9667	-.0242	.9400	-.0510
25.0	.9935	.9887	-.0047	.9745	-.0190	.9532	-.0403

TABLE 6.3.1 (cont'd.)

U	$\bar{F}_4(U)$	$\bar{\beta}_1^2 = 0$		$\bar{\beta}_1^2 = .25$		$\bar{\beta}_1^2 = 1$	
		$F_4(U)$	$F_4(U)-\bar{F}_4(U)$	$F_4(U)$	$F_4(U)-\bar{F}_4(U)$	$F_4(U)$	$F_4(U)-\bar{F}_4(U)$
26.0	.9953	.9918	-.0035	.9806	-.0147	.9638	-.0315
27.0	.9958	.9932	-.0026	.9845	-.0113	.9714	-.0243
28.0	.9969	.9950	-.0019	.9883	-.0086	.9783	-.0186
29.0	.9977	.9963	-.0013	.9912	-.0065	.9835	-.0141
30.0	.9974	.9965	-.0010	.9926	-.0048	.9868	-.0106
31.0	.9980	.9973	-.0007	.9944	-.0036	.9901	-.0080
32.0	.9985	.9980	-.0005	.9958	-.0027	.9926	-.0059
33.0	.9979	.9976	-.0003	.9960	-.0019	.9936	-.0043
34.0	.9984	.9981	-.0002	.9969	-.0014	.9952	-.0032
35.0	.9987	.9985	-.0002	.9977	-.0010	.9964	-.0023

TABLE 6.3.2

$F_4(U)$: $\nu = 1$, $m = 10$, $\bar{\mu}^2 = 10$

U	$\bar{F}_4(U)$	$\bar{\beta}_1^2 = 0$		$\bar{\beta}_1^2 = .25$		$\bar{\beta}_1^2 = 1$	
		$F_4(U)$	$F_4(U)-\bar{F}_4(U)$	$F_4(U)$	$F_4(U)-\bar{F}_4(U)$	$F_4(U)$	$F_4(U)-\bar{F}_4(U)$
1.0	.0001	.0000	-.0000	.0001	.0000	.0002	.0001
2.0	.0015	.0009	-.0006	.0022	.0007	.0041	.0026
3.0	.0093	.0057	-.0036	.0123	.0030	.0223	.0130
4.0	.0301	.0194	-.0106	.0371	.0070	.0636	.0335
5.0	.0688	.0468	-.0220	.0793	.0105	.1281	.0593
6.0	.1265	.0901	-.0364	.1371	.0106	.2077	.0812
7.0	.2008	.1492	-.0515	.2063	.0055	.2919	.0912
8.0	.2866	.2214	-.0651	.2815	-.0050	.3717	.0852
9.0	.3777	.3021	-.0755	.3579	-.0198	.4416	.0639
10.0	.4692	.3874	-.0818	.4327	-.0365	.5007	.0315
11.0	.5564	.4726	-.0838	.5035	-.0528	.5499	-.0064
12.0	.6355	.5535	-.0820	.5686	-.0669	.5914	-.0442
13.0	.7060	.6287	-.0772	.6287	-.0772	.6287	-.0772
14.0	.7664	.6960	-.0704	.6830	-.0834	.6635	-.1029
15.0	.8163	.7538	-.0624	.7308	-.0855	.6962	-.1201
16.0	.8578	.8038	-.0540	.7739	-.0839	.7291	-.1288
17.0	.8913	.8456	-.0457	.8118	-.0795	.7612	-.1301
18.0	.9177	.8798	-.0380	.8447	-.0730	.7921	-.1256
19.0	.9376	.9066	-.0310	.8723	-.0653	.8207	-.1169
20.0	.9536	.9287	-.0249	.8964	-.0572	.8481	-.1055
21.0	.9658	.9460	-.0198	.9168	-.0490	.8730	-.0928
22.0	.9741	.9586	-.0155	.9328	-.0412	.8942	-.0799
23.0	.9811	.9691	-.0120	.9469	-.0342	.9137	-.0674
24.0	.9863	.9771	-.0092	.9584	-.0279	.9304	-.0559
25.0	.9892	.9822	-.0070	.9667	-.0225	.9435	-.0457

TABLE 6.3.2 (cont'd.)

U	$\bar{F}_4(U)$	$\bar{\beta}_1^{\,2} = 0$		$\bar{\beta}_1^{\,2} = .25$		$\bar{\beta}_1^{\,2} = 1$	
		$F_4(U)$	$F_4(U)-\bar{F}_4(U)$	$F_4(U)$	$F_4(U)-\bar{F}_4(U)$	$F_4(U)$	$F_4(U)-\bar{F}_4(U)$
26.0	.9921	.9869	-.0053	.9742	-.0179	.9553	-.0368
27.0	.9943	.9904	-.0039	.9802	-.0141	.9650	-.0293
28.0	.9949	.9920	-.0029	.9840	-.0110	.9719	-.0230
29.0	.9962	.9941	-.0021	.9878	-.0085	.9783	-.0179
30.0	.9972	.9956	-.0016	.9907	-.0065	.9834	-.0138
31.0	.9970	.9958	-.0011	.9920	-.0049	.9864	-.0106
32.0	.9977	.9968	-.0008	.9940	-.0037	.9896	-.0080
33.0	.9982	.9976	-.0006	.9954	-.0028	.9922	-.0060
34.0	.9976	.9972	-.0004	.9956	-.0021	.9931	-.0045
35.0	.9981	.9978	-.0003	.9966	-.0015	.9948	-.0033

TABLE 6.3.3

$F_4(U)$: $\nu = 2$, $m = 10$, $\bar{\mu}^2 = 10$

U	$\bar{F}_4(U)$	$\bar{\beta}_1^2 = 0$		$\bar{\beta}_1^2 = .25$		$\bar{\beta}_1^2 = 1$	
		$F_4(U)$	$F_4(U)-\bar{F}_4(U)$	$F_4(U)$	$F_4(U)-\bar{F}_4(U)$	$F_4(U)$	$F_4(U)-\bar{F}_4(U)$
1.0	.0000	.0000	-.0000	.0000	.0000	.0001	.0000
2.0	.0006	.0003	-.0003	.0010	.0004	.0019	.0013
3.0	.0045	.0027	-.0018	.0068	.0023	.0129	.0085
4.0	.0166	.0105	-.0060	.0235	.0069	.0429	.0264
5.0	.0420	.0281	-.0139	.0557	.0137	.0971	.0551
6.0	.0838	.0586	-.0252	.1043	.0205	.1728	.0889
7.0	.1423	.1038	-.0385	.1670	.0247	.2618	.1195
8.0	.2146	.1625	-.0521	.2390	.0244	.3538	.1392
9.0	.2968	.2328	-.0641	.3159	.0190	.4405	.1437
10.0	.3838	.3107	-.0731	.3930	.0092	.5165	.1326
11.0	.4705	.3919	-.0786	.4669	-.0036	.5793	.1089
12.0	.5538	.4735	-.0803	.5364	-.0174	.6307	.0769
13.0	.6305	.5518	-.0787	.5999	-.0306	.6721	.0416
14.0	.6983	.6238	-.0745	.6566	-.0417	.7058	.0075
15.0	.7578	.6894	-.0684	.7078	-.0500	.7353	-.0225
16.0	.8081	.7470	-.0611	.7530	-.0551	.7619	-.0462
17.0	.8491	.7958	-.0533	.7919	-.0572	.7859	-.0632
18.0	.8833	.8377	-.0455	.8265	-.0567	.8097	-.0735
19.0	.9106	.8724	-.0382	.8564	-.0541	.8326	-.0781
20.0	.9313	.8998	-.0315	.8812	-.0501	.8534	-.0779
21.0	.9483	.9227	-.0256	.9032	-.0451	.8740	-.0743
22.0	.9614	.9409	-.0205	.9218	-.0397	.8930	-.0684
23.0	.9714	.9552	-.0163	.9372	-.0342	.9103	-.0611
24.0	.9782	.9654	-.0127	.9492	-.0290	.9249	-.0533
25.0	.9840	.9741	-.0099	.9598	-.0241	.9384	-.0456

TABLE 6.3.3 (cont'd.)

U	$\bar{F}_4(U)$	$\bar{\beta}_1^2 = 0$		$\bar{\beta}_1^2 = .25$		$\bar{\beta}_1^2 = 1$	
		$F_4(U)$	$F_4(U)-\bar{F}_4(U)$	$F_4(U)$	$F_4(U)-\bar{F}_4(U)$	$F_4(U)$	$F_4(U)-\bar{F}_4(U)$
26.0	.9883	.9807	-.0076	.9685	-.0198	.9501	-.0382
27.0	.9907	.9849	-.0058	.9746	-.0161	.9591	-.0315
28.0	.9932	.9888	-.0043	.9803	-.0129	.9675	-.0257
29.0	.9950	.9917	-.0033	.9848	-.0102	.9744	-.0206
30.0	.9955	.9931	-.0024	.9875	-.0080	.9791	-.0164
31.0	.9966	.9949	-.0018	.9904	-.0062	.9838	-.0128
32.0	.9975	.9962	-.0013	.9927	-.0048	.9875	-.0100
33.0	.9973	.9963	-.1010	.9936	-.0037	.9896	-.0077
34.0	.9979	.9972	-.0007	.9951	-.0028	.9920	-.0059
35.0	.9984	.9979	-.0005	.9963	-.0021	.9939	-.0045

TABLE 6.3.4

$F_4(U)$: $\nu = 3$, $m = 10$, $\bar{\mu}^2 = 10$

U	$\bar{F}_4(U)$	$\bar{\beta}_1^2 = 0$		$\bar{\beta}_1^2 = .25$		$\bar{\beta}_1^2 = 1$	
		$F_4(U)$	$F_4(U)-\bar{F}_4(U)$	$F_4(U)$	$F_4(U)-\bar{F}_4(U)$	$F_4(U)$	$F_4(U)-\bar{F}_4(U)$
1.0	.0000	.0000	-.0000	.0000	.0000	.0000	.0000
2.0	.0002	.0001	-.0001	.0004	.0002	.0009	.0006
3.0	.0021	.0012	-.0008	.0036	.0015	.0071	.0051
4.0	.0088	.0055	-.0033	.0143	.0055	.0275	.0187
5.0	.0248	.0163	-.0085	.0376	.0129	.0697	.0449
6.0	.0538	.0370	-.0168	.0767	.0228	.1361	.0823
7.0	.0978	.0701	-.0277	.1311	.0333	.2227	.1249
8.0	.1563	.1162	-.0401	.1979	.0417	.3206	.1644
9.0	.2269	.1746	-.0523	.2729	.0460	.4204	.1935
10.0	.3057	.2428	-.0629	.3510	.0453	.5132	.2076
11.0	.3888	.3179	-.0709	.4285	.0397	.5944	.2056
12.0	.4721	.3964	-.0757	.5023	.0303	.6613	.1892
13.0	.5515	.4742	-.0772	.5700	.0185	.7137	.1622
14.0	.6256	.5497	-.0758	.6316	.0061	.7545	.1289
15.0	.6922	.6201	-.0720	.6864	-.0058	.7858	.0936
16.0	.7498	.6834	-.0665	.7339	-.0160	.8097	.0598
17.0	.7999	.7401	-.0598	.7759	-.0239	.8297	.0299
18.0	.8418	.7892	-.0526	.8124	-.0294	.8471	.0053
19.0	.8756	.8303	-.0453	.8430	-.0326	.8620	-.0137
20.0	.9037	.8654	-.0384	.8700	-.0338	.8769	-.0269
21.0	.9262	.8943	-.0320	.8930	-.0332	.8911	-.0351
22.0	.9431	.9169	-.0262	.9116	-.0315	.9038	-.0393
23.0	.9571	.9358	-.0212	.9282	-.0289	.9167	-.0403
24.0	.9679	.9509	-.0170	.9420	-.0258	.9288	-.0391
25.0	.9752	.9618	-.0134	.9526	-.0226	.9389	-.0363

TABLE 6.3.4 (cont'd.)

U	$\bar{F}_4(U)$	$\bar{\beta}_1^2 = 0$		$\bar{\beta}_1^2 = .25$		$\bar{\beta}_1^2 = 1$	
		$F_4(U)$	$F_4(U)-\bar{F}_4(U)$	$F_4(U)$	$F_4(U)-\bar{F}_4(U)$	$F_4(U)$	$F_4(U)-\bar{F}_4(U)$
26.0	.9816	.9711	-.0105	.9623	-.0193	.9490	-.0326
27.0	.9864	.9783	-.0081	.9701	-.0163	.9579	-.0285
28.0	.9891	.9828	-.0063	.9756	-.0135	.9647	-.0244
29.0	.9920	.9872	-.0048	.9809	-.0111	.9714	-.0205
30.0	.9941	.9905	-.0036	.9851	-.0089	.9771	-.0169
31.0	.9947	.9920	-.0027	.9875	-.0071	.9809	-.0138
32.0	.9960	.9940	-.0020	.9904	-.0057	.9849	-.0111
33.0	.9970	.9955	-.0015	.9926	-.0044	.9882	-.0088
34.0	.9968	.9957	-.0011	.9934	-.0034	.9899	-.0069
35.0	.9975	.9967	-.0008	.9949	-.0026	.9921	-.0054

TABLE 6.3.5

$F_4(U)$: $\nu = 4$, $m = 10$, $\bar{\mu}^2 = 10$

U	$\bar{F}_4(U)$	$\bar{\beta}_1^2 = 0$		$\bar{\beta}_1^2 = .25$		$\bar{\beta}_1^2 = 1$	
		$F_4(U)$	$F_4(U)-\bar{F}_4(U)$	$F_4(U)$	$F_4(U)-\bar{F}_4(U)$	$F_4(U)$	$F_4(U)-\bar{F}_4(U)$
1.0	.0000	.0000	-.0000	.0000	.0000	.0000	.0000
2.0	.0001	.0000	-.0000	.0002	.0001	.0004	.0003
3.0	.0009	.0005	-.0004	.0018	.0009	.0038	.0028
4.0	.0045	.0028	-.0017	.0084	.0038	.0167	.0122
5.0	.0142	.0092	-.0050	.0246	.0104	.0476	.0334
6.0	.0335	.0227	-.0108	.0544	.0210	.1021	.0686
7.0	.0652	.0459	-.0193	.0997	.0345	.1804	.1152
8.0	.1106	.0808	-.0298	.1594	.0488	.2773	.1667
9.0	.1689	.1277	-.0412	.2302	.0614	.3840	.2152
10.0	.2376	.1854	-.0522	.3076	.0700	.4909	.2533
11.0	.3138	.2520	-.0617	.3872	.0735	.5900	.2762
12.0	.3932	.3243	-.0688	.4648	.0716	.6754	.2822
13.0	.4730	.3999	-.0731	.5381	.0651	.7454	.2724
14.0	.5499	.4754	-.0745	.6051	.0551	.7995	.2496
15.0	.6211	.5478	-.0732	.6643	.0432	.8389	.2179
16.0	.6860	.6162	-.0698	.7167	.0307	.8674	.1815
17.0	.7433	.6786	-.0647	.7621	.0188	.8873	.1440
18.0	.7922	.7336	-.0586	.8004	.0082	.9005	.1083
19.0	.8342	.7823	-.0519	.8337	-.0005	.9107	.0765
20.0	.8692	.8241	-.0450	.8620	-.0072	.9187	.0495
21.0	.8971	.8586	-.0384	.8852	-.0119	.9249	.0279
22.0	.9203	.8881	-.0323	.9055	-.0149	.9316	.0113
23.0	.9389	.9122	-.0267	.9226	-.0163	.9382	-.0007
24.0	.9527	.9309	-.0218	.9361	-.0166	.9440	-.0087
25.0	.9642	.9466	-.0176	.9482	-.0160	.9506	-.0137

TABLE 6.3.5 (cont'd.)

U	$\bar{F}_4(U)$	$\bar{\beta}_1^2 = 0$		$\bar{\beta}_1^2 = .25$		$\bar{\beta}_1^2 = 1$	
		$F_4(U)$	$F_4(U)-\bar{F}_4(U)$	$F_4(U)$	$F_4(U)-\bar{F}_4(U)$	$F_4(U)$	$F_4(U)-\bar{F}_4(U)$
26.0	.9731	.9591	-.0141	.9582	-.0149	.9570	-.0162
27.0	.9791	.9680	-.0111	.9657	-.0134	.9622	-.0170
28.0	.9845	.9758	-.0087	.9726	-.0118	.9679	-.0165
29.0	.9885	.9817	-.0067	.9783	-.0102	.9731	-.0153
30.0	.9907	.9855	-.0052	.9820	-.0086	.9769	-.0137
31.0	.9931	.9891	-.0040	.9859	-.0072	.9811	-.0120
32.0	.9949	.9919	-.0030	.9890	-.0059	.9847	-.0102
33.0	.9953	.9931	-.0023	.9906	-.0048	.9868	-.0085
34.0	.9965	.9948	-.0017	.9927	-.0038	.9895	-.0070
35.0	.9974	.9961	-.0013	.9943	-.0030	.9917	-.0057

TABLE 6.3.6

$F_4(U)$: $\nu = 5$, $m = 10$, $\bar{\mu}^2 = 10$

U	$\bar{F}_4(U)$	$\bar{\beta}_1^2 = 0$		$\bar{\beta}_1^2 = .25$		$\bar{\beta}_1^2 = 1$	
		$F_4(U)$	$F_4(U)-\bar{F}_4(U)$	$F_4(U)$	$F_4(U)-\bar{F}_4(U)$	$F_4(U)$	$F_4(U)-\bar{F}_4(U)$
1.0	.0000	.0000	-.0000	.0000	.0000	.0000	.0000
2.0	.0000	.0000	-.0000	.0001	.0000	.0002	.0001
3.0	.0004	.0002	-.0002	.0009	.0005	.0019	.0015
4.0	.0023	.0014	-.0009	.0047	.0025	.0098	.0075
5.0	.0079	.0050	-.0028	.0155	.0076	.0311	.0232
6.0	.0202	.0135	-.0067	.0374	.0172	.0732	.0530
7.0	.0423	.0294	-.0129	.0735	.0312	.1398	.0975
8.0	.0762	.0548	-.0214	.1247	.0485	.2296	.1534
9.0	.1223	.0910	-.0314	.1891	.0668	.3364	.2141
10.0	.1801	.1382	-.0419	.2635	.0834	.4515	.2713
11.0	.2471	.1951	-.0520	.3431	.0960	.5651	.3180
12.0	.3207	.2601	-.0606	.4240	.1033	.6697	.3491
13.0	.3974	.3305	-.0669	.5022	.1047	.7597	.3623
14.0	.4739	.4031	-.0708	.5746	.1008	.8320	.3581
15.0	.5481	.4761	-.0720	.6405	.0924	.8871	.3390
16.0	.6169	.5460	-.0709	.6978	.0809	.9255	.3086
17.0	.6803	.6125	-.0678	.7480	.0678	.9513	.2710
18.0	.7366	.6736	-.0631	.7908	.0542	.9667	.2301
19.0	.7850	.7276	-.0574	.8262	.0412	.9740	.1891
20.0	.8270	.7759	-.0511	.8565	.0295	.9774	.1504
21.0	.8623	.8176	-.0447	.8818	.0195	.9781	.1158
22.0	.8907	.8522	-.0385	.9020	.0113	.9766	.0860
23.0	.9146	.8821	-.0326	.9196	.0050	.9759	.0613
24.0	.9339	.9067	-.0272	.9342	.0003	.9755	.0416
25.0	.9483	.9260	-.0224	.9454	-.0029	.9747	.0263

TABLE 6.3.6 (cont'd.)

U	$\bar{F}_4(U)$	$\bar{\beta}_1^2 = 0$		$\bar{\beta}_1^2 = .25$		$\bar{\beta}_1^2 = 1$	
		$F_4(U)$	$F_4(U)-\bar{F}_4(U)$	$F_4(U)$	$F_4(U)-\bar{F}_4(U)$	$F_4(U)$	$F_4(U)-\bar{F}_4(U)$
26.0	.9606	.9424	-.0182	.9557	-.0049	.9756	.0150
27.0	.9701	.9555	-.0147	.9641	-.0060	.9770	.0069
28.0	.9766	.9649	-.0117	.9701	-.0065	.9780	.0014
29.0	.9824	.9732	-.0092	.9760	-.0064	.9802	-.0022
30.0	.9869	.9796	-.0072	.9808	-.0060	.9826	-.0042
31.0	.9893	.9837	-.0056	.9838	-.0055	.9840	-.0053
32.0	.9920	.9877	-.0043	.9872	-.0048	.9864	-.0056
33.0	.9940	.9907	-.0033	.9899	-.0041	.9886	-.0054
34.0	.9946	.9921	-.0025	.9911	-.0035	.9896	-.0050
35.0	.9959	.9940	-.0019	.9930	-.0029	.9915	-.0044

TABLE 6.3.7

$F_4(U^*)-\bar{F}_4(U^*)$: $m = 10$, $\bar{\mu}^2 = 10$

$\bar{\beta}_1^2$	$\nu = 0$		$\nu = 1$		$\nu = 2$		$\nu = 3$	
	U*	$F_4(U^*)-\bar{F}_4(U^*)$	U*	$F_4(U^*)-\bar{F}_4(U^*)$	U*	$F_4(U^*)-\bar{F}_4(U^*)$	U*	$F_4(U^*)-\bar{F}_4(U^*)$
0	10.0000	-.0877	11.0000	-.0838	12.0000	-.0803	13.0000	-.0772
.25	3.7830	.0037	5.5481	.0111	7.4540	.0252	9.3590	.0463
.25	13.2170	-.1164	15.0353	-.0855	17.2793	-.0573	20.1410	-.0338
1	5.5279	.0524	7.1375	.0913	8.7800	.1441	10.3688	.2087
1	14.4721	-.1926	16.6958	-.1304	19.4534	-.0785	22.8812	-.0404

TABLE 6.3.7 (cont'd.)

$\bar{\beta}_1^2$	$\nu = 4$		$\nu = 5$	
	U*	$F_4(U^*)-\bar{F}_4(U^*)$	U*	$F_4(U^*)-\bar{F}_4(U^*)$
0	14.0000	-.0745	15.0000	-.0720
.25	11.1407	.0735	12.7500	.1049
.25	23.7926	-.0166	28.3333	-.0065
1	11.8585	.2824	13.2444	.3628
1	27.0748	-.0170	32.0889	-.0056

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APPENDIX

APPENDIX

MATHEMATICAL APPENDIX ON SPECIAL FUNCTIONS

This appendix consists of a few of the definitions and theorems from special function theory which have been found to be of value in the course of deriving and tabulating exact finite sample distribution functions. In the case of established definitions and theorems the sources are cited, whereas the proofs of original theorems are given.

The factorial function, $(\alpha)_n$, (Rainville, 1960, p. 22) is defined by

$$\text{(A.1)} \quad (\alpha)_n = \prod_{k=1}^{n} (\alpha + k - 1) \quad , \quad n \geqq 1$$

$$(\alpha)_o = 1 \, , \qquad \alpha \neq 0 \, .$$

If α is neither zero nor a negative integer (Rainville, 1960, p. 23),

$$\text{(A.2)} \quad (\alpha)_n = \Gamma(\alpha + n)/\Gamma(\alpha).$$

Legendre's duplication formula (Rainville, 1960, p. 24) is given by

$$\text{(A.3)} \quad \sqrt{\pi} \;\; \Gamma(2z) = 2^{2z-1} \, \Gamma(z) \, \Gamma(z + \tfrac{1}{2})$$

The generalized hypergeometric function (Rainville, 1960, p. 73) is defined by

$$\text{(A.4)}\qquad {}_pF_q\left[\begin{matrix}\alpha_1, \alpha_2, \ldots, \alpha_p;\\ \beta_1, \beta_2, \ldots, \beta_q;\end{matrix}\; z\right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \cdot \frac{z^n}{n!}$$

where no denominator parameter, β_j, may be zero or a negative integer and if any numerator parameter, α_i, is zero or a negative integer, the series terminates. Furthermore,

(a) If $p \leqq q$, the series converges for all finite z;

(b) If $p = q + 1$, the series converges for $|z| < 1$ and diverges for $|z| > 1$;

(c) If $p > q + 1$, the series diverges for $z \neq 0$.

If the series terminates, there is no question of convergence, and (b) and (c) do not apply. If $p = q + 1$ the series is absolutely convergent for $z = \pm 1$ if $\sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i > 0$.

For $p \leqq 2$ and $q \leqq 1$ a slightly modified notation will be used, e.g.

$${}_1F_0\ (\alpha_1; - ; z) = {}_1F_0\left[\begin{matrix}\alpha_1\ ;\\ -\ ;\end{matrix}\; z\right] .$$

If c-a-b > 0 and if c is neither zero nor a negative integer then (Rainville, 1960, p. 49)

$$\text{(A.5)}\qquad {}_2F_1\ (a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} .$$

The ordinary binomial expansion (Rainville, 1960, p. 58) is given by

(A.6) $(1 - z)^{-a} = {}_1F_0\,(a; - ; z)$.

We have (Rainville, 1960, p. 58)

(A.7a-b) $(\alpha)_{n-k} = (-1)^k (\alpha)_n / (1-\alpha-n)_k$
$0 \leqq k \leqq n$

$(n-k)! = (-1)^k\, n!/(-n)_k$
$0 \leqq k \leqq n$.

We obtain the following theorems (Luke, 1969, Vol. I, p. 99):

(A.8) ${}_2F_1\,(-n, b; c; 1) = (c-b)_n/(c)_n$

where n is zero or a positive integer,

(A.9) ${}_2F_1\,(-n, n+\lambda; c; 1) = (-1)^n (1+\lambda-c)_n/(c)_n$

where n is zero or a positive integer, and when c is a negative integer or zero, c = -m, with m > n, then Vandermonde's theorem is given by

$$\text{(A.10)} \quad \sum_{k=0}^{n} \frac{(-n)_k (b)_k}{(-m)_k \; k!} = \frac{(m-n)!(m+b+1-n)_n}{m!}$$

We have (Luke, 1969, Vol. II, pp.64,65)

$$\text{(A.11)} \quad {}_{p+2}F_{q+1}\left[\begin{matrix} \sigma_1, \sigma_2, \alpha_1, \ldots, \alpha_p; \\ \delta, \rho_1, \ldots, \rho_q; \end{matrix} \; z\right] =$$

$$\sum_{n=0}^{\infty} \frac{(\delta-\sigma_1)_n (\delta-\sigma_2)_n (\alpha_1)_n \cdots (\alpha_p)_n}{(\delta)_n (\rho_1)_n \cdots (\rho_q)_n} \cdot \frac{z^n}{n!}$$

$$\cdot \; {}_{p+1}F_q\left[\begin{matrix} \sigma_1+\sigma_2-\delta, \alpha_1+n, \ldots, \alpha_p+n; \\ \rho_1+n, \ldots, \rho_q+n; \end{matrix} \; z\right]$$

$p < q$ or $p = q$ and $|z| < 1$,

$$\text{(A.12)} \quad {}_{p+2}F_{q+1}\left[\begin{matrix} \sigma_1, \sigma_2, \alpha_1, \ldots, \alpha_p; \\ \delta, \rho_1, \ldots, \rho_q; \end{matrix} \; z\right] =$$

$$\sum_{n=0}^{\infty} \frac{(\sigma_1+\sigma_2-\delta)_n (\alpha_1)_n \cdots (\alpha_p)_n}{(\rho_1)_n \cdots (\rho_q)_n} \cdot \frac{z^n}{n!}$$

$$\cdot \; {}_{p+2}F_{q+1}\left[\begin{matrix} \delta-\sigma_1, \delta-\sigma_2, \alpha_1+n, \ldots, \alpha_p+n; \\ \delta, \rho_1+n, \ldots, \rho_q+n; \end{matrix} \; z\right]$$

$p < q$ or $p = q$ and $|z| < 1$, and

$$\text{(A.13)}\quad {}_{p+1}F_{q+1}\left[\begin{matrix}\sigma+\delta,\ \alpha_1,\ \ldots,\ \alpha_p; \\ \delta,\ \rho_1,\ \ldots,\ \rho_q;\end{matrix}\ z\right] =$$

$$\sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\rho_1)_n \cdots (\rho_q)_n} \cdot \frac{z^n}{n!}$$

$$\cdot\ {}_{p+1}F_{q+1}\left[\begin{matrix}-\sigma,\ \alpha_1+n,\ \ldots,\ \alpha_p+n; \\ \delta,\ \rho_1+n,\ \ldots,\ \rho_q+n;\end{matrix}\ -z\right]$$

$p \leqq q$ or $p = q+1$ and $|z| < \frac{1}{2}$.

Notice that Kummer's first formula (Rainville, 1960, p. 125) is a special case of (A.13). Also, (Rainville, 1960, p. 60), if $|z| < 1$ and $|z/(1-z)| < 1$

$$\text{(A.14)}\quad {}_2F_1\ (a,\ b;\ c;\ z) = (1-z)^{-a}\ {}_2F_1\ (a,\ c-b;\ c;\ -z/(1-z)).$$

We have (Bateman, 1953, Vol. 1, p. 283)

$$\text{(A.15)}\quad {}_1F_1\ (a;\ b;\ xy) =$$

$$\sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \cdot \frac{[x(y-1)]^n}{n!} \cdot {}_1F_1\,(a+n;\ b+n;\ x) \quad \text{and}$$

(A.16) $${}_1F_1\,(a;\ b;\ xy) =$$

$$y^{-a} \sum_{n=0}^{\infty} \frac{(a)_n (1-y^{-1})^n}{n!} \quad {}_1F_1\,(a+n;\ b;\ x) \text{ if } y > \frac{1}{2}\,.$$

Also (Bateman, 1953, Vol. 1, p. 288)

(A.17) $$\sum_{n=0}^{\infty} \frac{(a)_n (b'-a')_n}{(b)_n (b')_n} \cdot \frac{z^n}{n!} \cdot {}_1F_1\,(a+n;\ b+n;\ x-z)$$

$$\cdot\ {}_1F_1(a';\ b'+n;\ y) = \sum_{n=0}^{\infty} \frac{(b-a)_n (a')_n}{(b)_n\ (b')_n} \cdot \frac{z^n}{n!}$$

$$\cdot\ {}_1F_1\,(a;\ b+n;\ x) \cdot {}_1F_1\,(a'+n;\ b'+n;\ y-z)$$

We have (Barnes, 1907, p. 62) for $x > 0$,

(A.18) $$\lim_{x\to\infty} e^{-x}\, x^{\sum_{i=1}^{q} \rho_i - \sum_{i=1}^{p} \alpha_i} \ {}_pF_q \begin{bmatrix} \alpha_1, \ldots, \alpha_p; & \\ & x \\ \rho_1, \ldots, \rho_q; & \end{bmatrix}$$

$$= \prod_{i=1}^{q} \Gamma(\rho_i) / \prod_{i=1}^{p} \Gamma(\alpha_i)$$

and (Luke, 1969, Vol. I, p. 128)

(A.19) $${}_1F_1\,(a;\, b;\, x) \doteq \frac{\Gamma(b)}{\Gamma(b-a)}\,(-x)^{-a}$$

$$\cdot\, {}_2F_0\,(a,\, 1+a-b;\, -\,;\, -x^{-1}) + \frac{\Gamma(b)}{\Gamma(a)}\, e^x\, x^{a-b}$$

$$\cdot\, {}_2F_0\,(b-a,\, 1-a;\, -\,;\, x^{-1})\ .$$

One of the <u>second order confluent hypergeometric functions</u> <u>$\Phi_2(\alpha, \alpha'; \gamma; x, y)$</u> (Bateman, 1953, Vol. 1, p. 225) is defined by

(A.20) $$\Phi_2\,(\alpha,\, \alpha';\, \gamma;\, x,\, y) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(\alpha)_r(\alpha')_s}{(\gamma)_{r+s}} \cdot \frac{x^r}{r!} \cdot \frac{y^s}{s!}$$

$\Phi_2(\alpha, \alpha'; \gamma; x, y)$ can be written (Humbert, 1921, p. 76) as

(A.21) $$\Phi_2(\alpha,\, \alpha';\, \gamma;\, x,\, y) = \sum_{r=0}^{\infty} \frac{(-x)^r}{r!} \cdot \frac{(\alpha)_r(\alpha')_r}{(\gamma)_r(\gamma-\alpha')_r}$$

$$\cdot\, {}_1F_1\,(\alpha+r;\, \gamma-\alpha'+r;\, x) \quad \cdot\, {}_1F_1\,(\alpha'+r;\, \gamma+r;\, y)$$

When $k > 0$, the following theorem holds:

(A.22) $$\lim_{x\to+\infty}\ x^{\delta}\, e^{-x}\, \Phi_2(\alpha,\, \alpha';\, \alpha+\delta;\, x,\, -kx) =$$

$$\frac{\Gamma(\alpha+\delta)}{\Gamma(\alpha)} \quad (1+k)^{-\alpha'}$$

Proof: By (A.21),

$$\lim_{x\to\infty} \quad x^{\delta}\, e^{-x}\, \Phi_2(\alpha,\ \alpha';\ \alpha+\delta;\ x,\ -kx) =$$

$$\lim_{x\to\infty} \quad x^{\delta}\, e^{-x} \sum_{r=0}^{\infty} \frac{(\alpha)_r(\alpha')_r}{(\alpha+\delta)_r(\alpha+\delta-\alpha')_r} \cdot \frac{(-x)^r}{r!}$$

$$\cdot \ {}_1F_1\ (\alpha+r;\ \alpha+\delta-\alpha'+r;\ x)$$

$$\cdot \ {}_1F_1\ (\alpha'+r;\ \alpha+\delta+r;\ -kx)$$

which, by (A.19) equals

$$\lim_{x\to\infty} \quad \frac{\Gamma(\alpha+\delta-\alpha')}{\Gamma(\alpha)} \quad x^{\alpha'} \sum_{r=0}^{\infty} \frac{(\alpha')_r}{(\alpha+\delta)_r} \cdot \frac{(-x)^r}{r!}$$

$$\cdot \ {}_1F_1\ (\alpha'+r;\ \alpha+\delta+r;\ -kx)$$

and by (A.15) this equals

$$\lim_{x\to\infty} \quad \frac{\Gamma(\alpha+\delta-\alpha')}{\Gamma(\alpha)} \quad x^{\alpha'} \quad {}_1F_1\ (\alpha';\ \alpha+\delta;\ -\ (1+k)x)$$

which, by (A.19), equals

$$\lim_{x\to\infty} \frac{\Gamma(\alpha+\delta-\alpha')}{\Gamma(\alpha)} \cdot x^{\alpha'} \cdot \frac{\Gamma(\alpha+\delta)}{\Gamma(\alpha+\delta-\alpha')} \, [(1+k)x]^{-\alpha'} =$$

$$\frac{\Gamma(\alpha+\delta)}{\Gamma(\alpha)} \cdot (1+k)^{-\alpha'} \qquad \text{Q.E.D.}$$

We have that

$$\text{(A.23)} \quad \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(\alpha)_r(\beta)_r(\beta')_s}{(\gamma)_r\ (\delta)_{r+s}} \cdot \frac{x^r}{r!} \cdot \frac{y^s}{s!} =$$

$$\sum_{t=0}^{\infty} \frac{(\gamma-\alpha)_t(\gamma-\beta)_t}{(\gamma)_t\ \ (\delta)_t} \cdot \frac{x^t}{t!} \ \ \Phi_2(\alpha+\beta-\gamma,\ \beta';\ \delta+t;\ x,\ y)$$

Proof: By (A.11),

$$\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(\alpha)_r(\beta)_r(\beta')_s}{(\gamma)_r\ (\delta)_{r+s}} \cdot \frac{x^r}{r!} \cdot \frac{y^s}{s!} =$$

$$\sum_{s=0}^{\infty} \frac{(\beta')_s}{(\delta)_s} \cdot \frac{y^s}{s!} \ \sum_{t=0}^{\infty} \frac{(\gamma-\alpha)_t(\gamma-\beta)_t}{(\gamma)_t\ (\delta+s)_t} \cdot \frac{x^t}{t!}$$

$$\cdot \sum_{q=0}^{\infty} \frac{(\alpha+\beta-\gamma)_q}{(\delta+t+s)_q} \cdot \frac{x^q}{q!} =$$

$$\sum_{t=0}^{\infty} \frac{(\gamma-\alpha)_t(\gamma-\beta)_t}{(\gamma)_t\ \ (\delta)_t} \cdot \frac{x^t}{t!} \ \sum_{q=0}^{\infty} \sum_{s=0}^{\infty} \frac{(\alpha+\beta-\gamma)_q(\beta')_s}{(\delta+t)_{s+q}} \cdot \frac{x^q}{q!} \cdot \frac{y^s}{s!}$$

which, by (A.20), equals

$$\sum_{t=0}^{\infty} \frac{(\gamma-\alpha)_t(\gamma-\beta)_t}{(\gamma)_t \quad (\delta)_t} \cdot \frac{x^t}{t!} \; \Phi_2(\alpha+\beta-\gamma,\ \beta';\ \delta+t;\ x,\ y) \qquad \text{Q.E.D.}$$

We define one of the <u>third order confluent hypergeometric functions</u> <u>$\Phi_2^3\ (\alpha,\ \alpha',\ \alpha'';\ \gamma;\ x,\ y,\ z)$</u> as

(A.24) $\Phi_2^3\ (\alpha,\ \alpha',\ \alpha'';\ \gamma;\ x,\ y,\ z) =$

$$\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \frac{(\alpha)_r(\alpha')_s(\alpha'')_t}{(\gamma)_{r+s+t}} \cdot \frac{x^r}{r!} \cdot \frac{y^s}{s!} \cdot \frac{z^t}{t!}$$

The following integral representations are introduced in order to derive the expansions for $\Phi_2^3\ (\alpha,\ \alpha',\ \alpha'';\ \gamma;\ x,\ y,\ z)$ corresponding to (A.21):

(A.25 a-c) ${}_1F_1\ (a;\ b;\ x) =$

$$\frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 e^{xu}\ u^{a-1}\ (1-u)^{b-a-1}\ du$$

$\Phi_2(\alpha,\ \alpha';\ \gamma;\ x,\ y) =$

$$\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\alpha')\Gamma(\gamma-\alpha-\alpha')} \int_0^1 \int_0^1 e^{xu+yv}\ u^{\alpha-1}\ v^{\alpha'-1}$$

$$(1-u-v)^{\gamma-\alpha-\alpha'-1}\ du\ dv$$

$$\Phi_2^3\,(\alpha,\ \alpha',\ \alpha'';\ \gamma;\ x,\ y,\ z) =$$

$$\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\alpha')\Gamma(\alpha'')\Gamma(\gamma-\alpha-\alpha'-\alpha'')} \int_0^1\int_0^1\int_0^1 e^{xu+yv+zw}$$

$$u^{\alpha-1}\, v^{\alpha'-1}\, w^{\alpha''-1}\, (1-u-v-w)^{\gamma-\alpha-\alpha'-\alpha''-1}\ du\ dv\ dw$$

The three integral representations above may be verified by writing a series expression for the exponential term each time u, v, or w appears in an exponent and performing the indicated integration. The method of proof in the following theorem is a generalization of a proof of (A.21) using (A.25 a,b) rather than the method indicated in Humbert's paper (Humbert, 1921, p. 76).

(A.26) $\Phi_2^3\,(\alpha,\ \alpha',\ \alpha'';\ \gamma;\ x,\ y,\ z) =$

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty} \frac{(-x)^{r+s}}{r!\ \ s!} \cdot \frac{(-y)^t}{t!} \cdot \frac{(\alpha)_{r+s}(\alpha')_{r+t}(\alpha'')_{s+t}(\gamma-\alpha'')_r}{(\gamma)_{r+s+t}(\gamma-\alpha'-\alpha'')_{r+s}(\gamma-\alpha'')_{r+t}}$$

$$\cdot\ {}_1F_1\,(\alpha+r+s;\ \gamma-\alpha'-\alpha''+r+s;\ x)$$

$$\cdot\ {}_1F_1\,(\alpha'+r+t;\ \gamma-\alpha''+r+t;\ y)$$

$$\cdot\ {}_1F_1\,(\alpha''+s+t;\ \gamma+r+s+t;\ z)$$

Proof: From (A.25c) and the following transformation:

$$w = p$$
$$v = n(1-p)$$
$$u = m(1-n)(1-p)$$

we can write Φ_2^3 $(\alpha, \alpha', \alpha''; \gamma; x, y, z) =$

$$\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\alpha')\Gamma(\alpha'')\Gamma(\gamma-\alpha-\alpha'-\alpha'')} \int_0^1\int_0^1\int_0^1 e^{xm+yn+zp}$$

$$\cdot\, m^{\alpha-1}\, n^{\alpha'-1}\, p^{\alpha''-1}\, (1-m)^{\gamma-\alpha-\alpha'-\alpha''-1}\, (1-n)^{\gamma-\alpha'-\alpha''-1}$$

$$\cdot\, (1-p)^{\gamma-\alpha''-1}\, e^{-n(1-p)xm-pxm-ynp}\, dm\, dn\, dp$$

Introducing a series expansion for the exponential term corresponding to each of the exponents -n(1-p)xm, - pxm, and -ynp we then have

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty} \frac{(-x)^{r+s}}{r!\ s!} \cdot \frac{(-y)^t}{t!} \cdot \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\alpha')\Gamma(\alpha'')\Gamma(\gamma-\alpha-\alpha'-\alpha'')}$$

$$\cdot \int_0^1 e^{xm}\, m^{\alpha+r+s-1}\, (1-m)^{\gamma-\alpha-\alpha'-\alpha''-1}\, dm$$

$$\cdot \int_0^1 e^{yn}\, n^{\alpha'+r+t-1}\, (1-n)^{\gamma-\alpha'-\alpha''-1}\, dn$$

$$\int_0^1 e^{zp}\, p^{\alpha''+s+t-1}\, (1-p)^{\gamma-\alpha''+r-1}\, dp$$

From (A.25a) and (A.2) this reduces to

$$\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \frac{(-x)^{r+s}}{r!\quad s!} \cdot \frac{(-y)^{t}}{t!} \cdot \frac{(\alpha)_{r+s}(\alpha')_{r+t}(\alpha'')_{s+t}(\gamma-\alpha'')_{r}}{(\gamma)_{r+s+t}(\gamma-\alpha'-\alpha'')_{r+s}(\gamma-\alpha'')_{r+t}}$$

$$\cdot \; {}_1F_1\;(\alpha+r+s;\; \gamma-\alpha'-\alpha''+r+s;\; x)$$

$$\cdot \; {}_1F_1\;(\alpha'+r+t;\; \gamma-\alpha''+r+t;\; y)$$

$$\cdot \; {}_1F_1\;(\alpha''+s+t;\; \gamma+r+s+t;\; z) \qquad \text{Q.E.D.}$$

An alternative expansion of $\Phi_2^3(\alpha, \alpha', \alpha''; \gamma; x, y, z)$ is given by

(A.27) $\Phi_2^3\;(\alpha, \alpha', \alpha''; \gamma; x, y, z) =$

$$\sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \frac{(-y)^{s}}{s!} \cdot \frac{(-z)^{t}}{t!} \cdot \frac{(\alpha)_{s+t}(\alpha')_{s}(\alpha'')_{t}}{(\gamma)_{s+t}(\gamma-\alpha)_{s+t}}$$

$$\cdot \; {}_1F_1\;(\alpha+s+t;\; \gamma+s+t;\; x)$$

$$\cdot \; \Phi_2\;(\alpha'+s,\; \alpha''+t;\; \gamma-\alpha+s+t;\; y, z)$$

Proof: From (A.25c) and the following transformation:

$$u = m$$
$$v = n(1-m)$$
$$w = p(1-m)$$

we can write $\Phi_2^3\;(\alpha, \alpha', \alpha''; \gamma; x, y, z) =$

$$\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\alpha')\Gamma(\alpha'')\Gamma(\gamma-\alpha-\alpha'-\alpha'')} \int_0^1 e^{xm}\, m^{\alpha-1}\, (1-m)^{\gamma-\alpha-1}\, dm$$

$$\cdot \int_0^1\int_0^1 e^{yn+zp}\, n^{\alpha'-1} p^{\alpha''-1}\, (1-n-p)^{\gamma-\alpha-\alpha'-\alpha''-1}$$

$$\cdot\, e^{-mny-mpz}\, dn\, dp$$

Introducing a series expansion for the exponential term corresponding to each of the exponents $-mny$ and $-mpz$ we have

$$\sum_{s=0}^{\infty}\sum_{t=0}^{\infty} \frac{(-y)^s}{s!} \cdot \frac{(-z)^t}{t!} \cdot \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\alpha')\Gamma(\alpha'')\Gamma(\gamma-\alpha-\alpha'-\alpha'')}$$

$$\cdot \int_0^1 e^{xm}\, m^{\alpha+s+t-1}\, (1-m)^{\gamma-\alpha-1}\, dm$$

$$\cdot \int_0^1\int_0^1 e^{yn+zp}\, n^{\alpha'+s-1}\, p^{\alpha''+t-1} (1-n-p)^{\gamma-\alpha-\alpha'-\alpha''-1}\, dn\, dp$$

From (A.25 a,b) and (A.2) this reduces to

$$\sum_{s=0}^{\infty}\sum_{t=0}^{\infty} \frac{(-y)^s}{s!} \cdot \frac{(-z)^t}{t!} \cdot \frac{(\alpha)_{s+t}(\alpha')_s(\alpha'')_t}{(\gamma)_{s+t}(\gamma-\alpha)_{s+t}}$$

$$\cdot\, {}_1F_1\, (\alpha+s+t;\ \gamma+s+t;\ x)$$

$$\cdot\, \Phi_2\, (\alpha'+s,\ \alpha''+t;\ \gamma-\alpha+s+t;\ y,\ z) \qquad \text{Q.E.D.}$$

When $k > 0$ the following theorem holds:

(A.28) $$\lim_{x\to+\infty} e^{-x}\, x^{\delta}\, \Phi_2^3\, (\alpha,\ \alpha',\ \alpha'';\ \alpha+\delta;\ x,\ -kx,\ -kx)$$

$$= \frac{\Gamma(\alpha+\delta)}{\Gamma(\alpha)} \quad (1+k)^{-\alpha'-\alpha''}$$

Proof: By (A.26),

$$\lim_{x\to\infty} \; e^{-x} \, x^{\delta} \; \Phi_2^3 \, (\alpha, \; \alpha', \; \alpha''; \; \alpha+\delta; \; x, \; -kx, \; -kx) =$$

$$\lim_{x\to\infty} \; e^{-x} \, x^{\delta} \; \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \frac{(-x)^{r+s}}{r! \quad s!} \cdot \frac{(kx)^t}{t!}$$

$$\cdot \frac{(\alpha)_{r+s}(\alpha')_{r+t}(\alpha'')_{s+t}(\alpha+\delta-\alpha'')_r}{(\alpha+\delta)_{r+s+t}(\alpha+\delta-\alpha'-\alpha'')_{r+s}(\alpha+\delta-\alpha'')_{r+t}}$$

$$\cdot \; {}_1F_1 \; (\alpha+r+s; \; \alpha+\delta-\alpha'-\alpha''+r+s; \; x)$$

$$\cdot \; {}_1F_1 \; (\alpha'+r+t; \; \alpha+\delta-\alpha''+r+t; \; -kx)$$

$$\cdot \; {}_1F_1 \; (\alpha''+s+t; \; \alpha+\delta+r+s+t; \; -kx)$$

which, by (A.19), equals

$$\lim_{x\to\infty} \; e^{-x} \, x^{\delta} \; \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-x)^{r+s}}{r! \quad s!} \cdot \frac{(\alpha)_{r+s}(\alpha')_r(\alpha'')_s}{(\alpha+\delta)_{r+s}(\alpha+\delta-\alpha'-\alpha'')_{r+s}}$$

$$\cdot \; {}_1F_1 \; (\alpha+r+s; \; \alpha+\delta-\alpha'-\alpha''+r+s; \; x)$$

$$\cdot \; {}_1F_1 \; (\alpha'+r; \; \alpha+\delta-\alpha''+r; \; -kx)$$

$$\cdot\ {}_1F_1\ (\alpha''+s;\ \alpha+\delta+r+s;\ -kx)$$

and by (A.19) this equals

$$\lim_{x\to\infty} x^{\alpha'+\alpha''} \cdot \frac{\Gamma(\alpha+\delta-\alpha'-\alpha'')}{\Gamma(\alpha)} \sum_{r=0}^{\infty} \frac{(-x)^r}{r!} \cdot \frac{(\alpha')_r}{(\alpha+\delta)_r}$$

$$\cdot\ {}_1F_1\ (\alpha'+r;\ \alpha+\delta-\alpha''+r;\ -kx) \sum_{s=0}^{\infty} \frac{(-x)^s}{s!} \cdot \frac{(\alpha'')_s}{(\alpha+\delta+r)_s}$$

$$\cdot\ {}_1F_1\ (\alpha''+s;\ \alpha+\delta+r+s;\ -kx)$$

which, by (A.15), equals

$$\lim_{x\to\infty} x^{\alpha'+\alpha''} \cdot \frac{\Gamma(\alpha+\delta-\alpha'-\alpha'')}{\Gamma(\alpha)} \sum_{r=0}^{\infty} \frac{(-x)^r}{r!} \cdot \frac{(\alpha')_r}{(\alpha+\delta)_r}$$

$$\cdot\ {}_1F_1\ (\alpha'+r;\ \alpha+\delta-\alpha''+r;\ -kx)$$

$$\cdot\ {}_1F_1\ (\alpha'';\ \alpha+\delta+r;\ -\ (1+k)x)$$

and by (A.17) this equals

$$\lim_{x\to\infty} x^{\alpha'+\alpha''} \cdot \frac{\Gamma(\alpha+\delta-\alpha'-\alpha'')}{\Gamma(\alpha)} \sum_{r=0}^{\infty} \frac{(-x)^r}{r!} \cdot \frac{(\alpha+\delta-\alpha'-\alpha'')_r(\alpha'')_r}{(\alpha+\delta-\alpha'')_r(\alpha+\delta)_r}$$

$$\cdot \ {}_1F_1\ (\alpha';\ \alpha+\delta-\alpha''+r;\ -\ (1+k)x)$$

$$\cdot \ {}_1F_1\ (\alpha''+r;\ \alpha+\delta+r;\ -kx)$$

which, by (A.19), equals

$$\lim_{x\to\infty} x^{\alpha''}\ (1+k)^{-\alpha'} \cdot \frac{\Gamma(\alpha+\delta-\alpha'')}{\Gamma(\alpha)} \quad \sum_{r=0}^{\infty} \frac{(-x)^r}{r!} \cdot \frac{(\alpha'')_r}{(\alpha+\delta)_r}$$

$$\cdot \ {}_1F_1\ (\alpha''+r;\ \alpha+\delta+r;\ -kx)$$

and by (A.15) this equals

$$\lim_{x\to\infty} x^{\alpha''}\ (1+k)^{-\alpha'} \cdot \frac{\Gamma(\alpha+\delta-\alpha'')}{\Gamma(\alpha)} \quad {}_1F_1\ (\alpha'';\ \alpha+\delta;\ -\ (1+k)x)$$

which, by (A.19), equals

$$\frac{\Gamma(\alpha+\delta)}{\Gamma(\alpha)} \cdot (1+k)^{-\alpha'-\alpha''} \qquad \text{Q.E.D.}$$

We have that

$$\text{(A.29)} \quad \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \frac{(\alpha)_r(\alpha')_s(\alpha'')_t(\beta)_r(\beta')_s(\beta'')_t}{(\gamma)_r(\gamma')_s(\gamma'')_t\ (\delta)_{r+s+t}} \cdot \frac{x^r y^s z^t}{r!s!t!} =$$

$$\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \frac{(\gamma-\alpha)_r(\gamma'-\alpha')_s(\gamma''-\alpha'')_t(\gamma-\beta)_r(\gamma'-\beta')_s(\gamma''-\beta'')_t}{(\gamma)_r(\gamma')_s(\gamma'')_t \qquad (\delta)_{r+s+t}}$$

$$\cdot \frac{x^r y^s z^t}{r!s!t!} \cdot \Phi_2^3 (\alpha+\beta-\gamma, \alpha'+\beta'-\gamma', \alpha''+\beta''-\gamma''; \delta+r+s+t; x,y,z)$$

Proof: The expansion follows directly by application of (A.11) to each of the three series in turn and then by use of the definition (A.24). Q.E.D.

While no present application of the following generalization is made it is introduced for completeness and it is seen that (A.21) and (A.27) are special cases of (A.32). We define one of the <u>kth order confluent hypergeometric functions</u> $\underline{\Phi_2^k (\alpha_1, \ldots, \alpha_k; \gamma; x_1, \ldots, x_k)}$ as

(A.30) $\Phi_2^k (\alpha_1, \ldots, \alpha_k; \gamma; x_1, \ldots, x_k) =$

$$\sum_{t_1=0}^{\infty} \cdots \sum_{t_k=0}^{\infty} \frac{(\alpha_1)_{t_1} \cdots (\alpha_k)_{t_k}}{(\gamma)_{t_1 + \ldots + t_k}} \cdot \frac{x_1^{t_1} \cdots x_k^{t_k}}{t_1! \cdots t_k!}$$

Notice that $\Phi_2^1 (\alpha_1; \gamma; x_1) = {}_1F_1 (\alpha_1; \gamma; x_1)$ and $\Phi_2^2 (\alpha_1, \alpha_2; \gamma; x_1, x_2) = \Phi_2 (\alpha_1, \alpha_2; \gamma; x_1, x_2)$. The following integral representation for $\Phi_2^k (\alpha_1, \ldots, \alpha_k; \gamma; x_1, \ldots, x_k)$ may be verified by writing a series expression for the exponential term each time $u_1, \ldots, u_k$ appears in an exponent and performing the indicated integration.

(A.31) $\Phi_2^k (\alpha_1, \ldots, \alpha_k; \gamma; x_1, \ldots, x_k) =$

$$\frac{\Gamma(\gamma)}{\Gamma(\alpha_1)\ \dots\ \Gamma(\alpha_k)\Gamma(\gamma-\alpha_1-\ \dots\ -\alpha_k)} \int_0^1 \dots \int_0^1 e^{x_1 u_1 + \dots + x_k u_k}$$

$$\cdot\ u_1^{\alpha_1 - 1}\ \dots\ u_k^{\alpha_k - 1}\ (1 - u_1 - \dots - u_k)^{\gamma-\alpha_1 - \dots - \alpha_k - 1}$$

$$\cdot\ du_1\ \dots\ du_k$$

The following theorem is valid for $k \geqq 2$:

(A.32) $\Phi_2^k\ (\alpha_1, \dots, \alpha_k; \gamma; x_1, \dots, x_k) =$

$$\sum_{p_1=0}^{\infty} \cdots \sum_{p_{k-1}=0}^{\infty} \frac{(-x_1)^{p_1}\ \dots\ (-x_{k-1})^{p_{k-1}}}{p_1!\ \dots\ p_{k-1}!}$$

$$\cdot\ \frac{(\alpha_1)_{p_1}\ \cdots\ (\alpha_{k-1})_{p_{k-1}} (\alpha_k)_{p_1+\ \dots\ +p_{k-1}}}{(\gamma)_{p_1+\ \dots\ +p_{k-1}}\ (\gamma-\alpha_k)_{p_1+\ \dots\ +p_{k-1}}}$$

$$\cdot\ \Phi_2^{k-1}\ (\alpha_1+p_1, \dots\ \alpha_{k-1}+p_{k-1};\ \gamma-\alpha_k+p_1+\ \dots\ +p_{k-1};\ x_1, \dots, x_{k-1})$$

$$\cdot\ \Phi_2^1\ (\alpha_k+p_1+\ \dots\ +p_{k-1};\ \gamma+p_1+\ \dots\ +p_{k-1};\ x_k)$$

Proof: From (A.31) and the following transformation:

$$u_1 = r_1(1-r_k)$$

$$\begin{array}{cc} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{array}$$

$$u_{k-1} = r_{k-1}(1-r_k)$$
$$u_k = r_k$$

we can write $\Phi_2^k\ (\alpha_1, \ldots, \alpha_k;\ \gamma;\ x_1, \ldots, x_k) =$

$$\frac{\Gamma(\gamma)}{\Gamma(\alpha_1)\ \ldots\ \Gamma(\alpha_k)\Gamma(\gamma-\alpha_1-\ \ldots\ -\alpha_k)} \int_0^1 \cdots \int_0^1 e^{x_1 r_1 +\ \ldots\ + x_{k-1} r_{k-1}}$$

$$\cdot\ r_1^{\alpha_1 - 1}\ \ldots\ r_{k-1}^{\alpha_{k-1}-1} (1-r_1-\ \ldots\ -r_{k-1})^{\gamma-\alpha_1-\ \ldots\ -\alpha_k - 1} .\ e^{x_k r_k}$$

$$\cdot\ r_k^{\alpha_k - 1} (1-r_k)^{\gamma-\alpha_k-1}\ e^{-x_1 r_1 r_k -\ \ldots\ -x_{k-1} r_{k-1} r_k}\ dr_1\ \ldots\ dr_k$$

Introducing a series expansion for the exponential term corresponding to each of the exponents $-x_1 r_1 r_k, \ldots, -x_{k-1} r_{k-1} r_k$ we have

$$\sum_{p_1=0}^{\infty} \cdots \sum_{p_{k-1}=0}^{\infty} \frac{(-x_1)^{p_1}\ \ldots\ (-x_{k-1})^{p_{k-1}}}{p_1!\ \ldots\ p_{k-1}!}$$

$$\cdot\ \frac{\Gamma(\gamma)}{\Gamma(\alpha_1)\ \ldots\ \Gamma(\alpha_k)\Gamma(\gamma-\alpha_1-\ \ldots\ -\alpha_k)}$$

$$\cdot \int_0^1 \cdots \int_0^1 e^{x_1 r_1 +\ \ldots\ + x_{k-1} r_{k-1}}\ r_1^{\alpha_1 + p_1 - 1}\ \ldots\ r_{k-1}^{\alpha_{k-1} + p_{k-1} - 1}$$

$$\cdot\ (1-r_1-\ \ldots\ -r_{k-1})^{\gamma-\alpha_1-\ \ldots\ -\alpha_k-1}\ dr_1\ \ldots\ dr_{k-1}$$

$$\cdot \int_0^1 e^{x_k r_k} r_k^{\alpha_k+p_1+\cdots+p_{k-1}-1} (1-r_k)^{\gamma-\alpha_k-1} dr_k$$

From (A.31) and (A.2) this reduces to

$$\sum_{p_1=0}^{\infty}\cdots\sum_{p_{k-1}=0}^{\infty} \frac{(-x_1)^{p_1}\cdots(-x_{k-1})^{p_{k-1}}}{p_1!\cdots p_{k-1}!}$$

$$\cdot \frac{(\alpha_1)_{p_1}\cdots(\alpha_{k-1})_{p_{k-1}}(\alpha_k)_{p_1+\cdots+p_{k-1}}}{(\gamma)_{p_1+\cdots+p_{k-1}}(\gamma-\alpha_k)_{p_1+\cdots+p_{k-1}}}$$

$$\cdot \Phi_2^{k-1}(\alpha_1+p_1, \ldots, \alpha_{k-1}+p_{k-1}; \gamma-\alpha_k+p_1+\cdots+p_{k-1}; x_1, \ldots, x_{k-1})$$

$$\cdot \Phi_2^1(\alpha_k+p_1+\cdots+p_{k-1}; \gamma+p_1+\cdots+p_{k-1}; x_k) \qquad \text{Q.E.D.}$$

The following theorem is valid for $k > 0$:

$$\text{(A.33)} \quad \lim_{x\to\infty} e^{-x} x^{\beta'+\delta-\beta-\gamma} \sum_{j=0}^{\infty}\sum_{q=0}^{\infty} \frac{x^j}{j!} \cdot \frac{(-kx)^q}{q!}$$

$$\cdot \frac{(\beta)_j(\gamma)_j(\alpha)_q}{(\beta')_j(\delta)_{j+q}} = \frac{\Gamma(\delta)\Gamma(\beta')}{\Gamma(\gamma)\Gamma(\beta)}(1+k)^{-\alpha}$$

Proof: By (A.2), (A.11), and (A.20) we have

$$\lim_{x\to\infty} e^{-x} x^{\beta'+\delta-\beta-\gamma} \sum_{j=0}^{\infty} \sum_{q=0}^{\infty} \frac{x^j}{j!} \cdot \frac{(-kx)^q}{q!} \cdot \frac{(\beta)_j(\gamma)_j(\alpha)_q}{(\beta')_j(\delta)_{j+q}} =$$

$$\lim_{x\to\infty} e^{-x} x^{\beta'+\delta-\beta-\gamma} \sum_{j=0}^{\infty} \frac{(\beta'-\beta)_j(\beta'-\gamma)_j}{(\beta')_j \quad (\delta)_j} \cdot \frac{x^j}{j!}$$

$$\cdot \Phi_2(\beta+\gamma-\beta', \alpha; \delta+j; x, -kx)$$

which by (A.5) and (A.22) equals

$$\frac{\Gamma(\delta)\Gamma(\beta')}{\Gamma(\gamma)\Gamma(\beta)} (1+k)^{-\alpha} \qquad \text{Q.E.D.}$$

The <u>incomplete beta function</u> is defined by

(A.34) $$I_B[a,b;x] = \int_0^x t^{a-1} (1-t)^{b-1} \, dt$$

The <u>beta function</u> is defined by

(A.35) $$B(a,b) = \int_0^1 t^{a-1} (1-t)^{b-1} \, dt$$

We have, (Bateman, 1953, Vol. 1, p. 87) and (A.11), (A.36 a-b) $$I_B[a,\ b;\ x] = \frac{x^a}{a}\ {}_2F_1\left[\begin{matrix} a,\ 1-b; \\ 1+a; \end{matrix}\ x\right]$$

for $|x| \leq 1$

$$I_B[a,\ b;\ x] = \frac{x^a(1-x)^b}{a}\ {}_2F_1\left[\begin{matrix} 1,\ a+b; \\ 1+a; \end{matrix}\ x\right]$$

for $|x| < 1$

Also, (Bateman, 1953, Vol. 1, p. 9)

(A.37) $$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

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